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Examples of Projective Resolution of Lengths 3n/4n That Do Not Satisfy Homological Properties of Nakayama Algebra

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Authors' contributions

This work was carried out in collaboration between all authors. Authors KBG and AA designed the study, performed and carried out the proofs, managed the literature searches as well as the analysis of the main work, and wrote the first draft of the manuscript.Author YDA gave paramount and insightful comments on all sections of the draft manuscript.All authors read and approved the final manuscript.

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Abstract

In [1], Gyamfi et al. described homological properties in relation to Nakayama Algebras with projectives that satisfied condition $Ext^n_\Lambda(M,N) = 0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N,M) = 0$ for $n \gg 0$, [1]. The purpose of this paper is to give a similar characterization of Nakayama algebras. In particular, we present Ext-groups of the Nakayama algebras with projectives that do not satisfy the condition $Ext^n_\Lambda(M,N) = 0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N,M) = 0$ for $n \gg 0$. To do this, we consider the Ext-groups of Nakayama algebra with projectives of lengths $3n$ and $4n$ using combinations of modules of different lengths.

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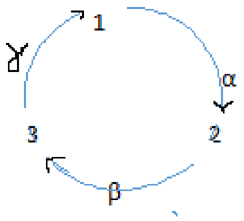
1 Introduction

In this section, we discuss some basic properties and some related examples of the Quivers, Path algebra, Projective resolutions and Ext-groups. A quiver is an oriented graph. In our discussions of quivers, we restrict ourselves to finite quivers. A quiver Γ is made up of a set of vertices, Γ_o, and a set of arrows, Γ_1 between these vertices. A quiver $\Gamma = (\Gamma_o, \Gamma_1)$.
Example: Let $\Gamma = 1 \longrightarrow 2 \longrightarrow 3$. Here there are three vertices 1, 2, and 3 which are joined by two arrows. This is represented diagrammatically by;



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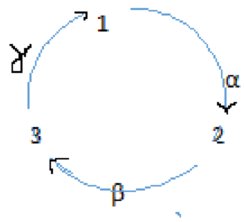
Let k be a field and $\Gamma = (\Gamma_0, \Gamma_1)$ be a quiver, then $k\Gamma$ is a $k-$vector space with the paths of Γ as the basis. This $k-$algebra $k\Gamma$, is called the path algebra of Γ over k. The elements in $k\Gamma$ are of the form $a_1p_1 + a_2p_2 + \cdots + a_np_n$, $a_i \in k$, and p_i a path in Γ. $\sum_{i\in\Gamma_0} e_i$ is the identity element of $k\Gamma$ [2]. For example; let $\Gamma = 1 \overset{\alpha}{\to} 2$, then a basis of this path algebra is e_1, e_2 and α. The elements in $k\Gamma$ are of the form; $a_1e_1 + a_2e_2 + a_3e_3$, $a_i \in k$ and $e_1, e_2, \alpha \in \Gamma$. We have the following table;

$x\backslash y$	e_1	e_2	α
e_1	e_1	0	0
e_2	0	e_2	α
α	α	0	0

This shows that , $(e_1 + e_2)(a_1e_1 + a_2e_2 + a_3\alpha) = a_1e_1 + a_2e_2 + a_3\alpha$. Hence $e_1 + e_2$ is the identity in $k\Gamma$.

Another example could as well be, let Γ be $\Gamma = 1 \overset{\alpha}{\to} 2 \overset{\beta}{\to} 3$, k a field, then a basis of $k\Gamma$ is $\{e_1, e_2, e_3, \alpha, \beta\alpha\}$. Then the dimension of $k\Gamma$ is given as; $dim_k k\Gamma = 5$. The identity in $k\Gamma$ is, $e_1 + e_2 + e_3 = 1_{k\Gamma}$.

Let M be a Λ-module. A projective resolution for M is an exact sequence $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ with the P_i projective modules for $i \geq 0$. A projective presentation $\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$ for M is called a minimal projective presentation if $f_0 : P_0 \to M$ and $f_i : P_i \to kerf_i$ are projective covers for $i \geq 1$ [2].

Example: Let $\Gamma =$ 

with the relation $\gamma\beta\alpha$, $\alpha\gamma\beta$, $\beta\alpha\gamma$ then the projectives are;

$$P_1 = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix} \; P_2 = \begin{pmatrix} S_2 \\ S_3 \\ S_1 \end{pmatrix} \; P_3 = \begin{pmatrix} S_3 \\ S_1 \\ S_2 \end{pmatrix}$$

where S_i is a simple module, $i = 1, 2, 3$. When we consider the projective P_1 as a representation, the module S_1 is in vertex one, S_2 is in vertex two and S_3 is in vertex three.

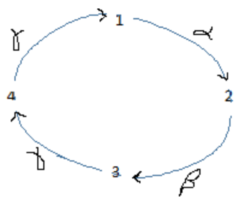
Finally, we discuss Ext-groups. Let M, N be Λ-modules.

Let $P \cdots \to P_3 \overset{d_3}{\to} P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0 \overset{d_0}{\to} M \to 0$ be the projective resolution for M. The following is the truncation of the exact sequence;

$P \cdots \to P_3 \overset{d_3}{\to} P_2 \overset{d_2}{\to} P_1 \overset{d_1}{\to} P_0$. Applying $Hom(\quad, N)$, we have;

$$0 \overset{d_0^{\times}}{\to} Hom_{\Lambda}(P_0, N) \overset{d_1^{\times}}{\to} Hom_{\Lambda}(P_1, N) \overset{d_2^{\times}}{\to} Hom_{\Lambda}(P_2, N) \qquad \overset{d_3^{\times}}{\to} Hom_{\Lambda}(P_3, N)$$

$Ext^i_{\Lambda}(M, N)$ is defined as; $Ext^i_{\Lambda}(M, N) = kerd_i^{\times}/Imd_{i-1}^{\times}$. eg $Ext^2_{\Lambda}(M, N) = kerd_2^{\times}/Imd_1^{\times}$, [5].

Example: Let $\Gamma =$ 

with relation $\delta\gamma\beta\alpha$, $\alpha\delta\gamma\beta$, $\beta\alpha\delta\gamma$ and $\gamma\beta\alpha\delta$ then we have the following projectives;

$$P_1 = \begin{pmatrix} S_1 \\ S_2 \\ S_3 \\ S_4 \end{pmatrix} \; P_2 = \begin{pmatrix} S_2 \\ S_3 \\ S_4 \\ S_1 \end{pmatrix} \; P_3 = \begin{pmatrix} S_3 \\ S_4 \\ S_1 \\ S_2 \end{pmatrix} \; P_4 = \begin{pmatrix} S_4 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

In our previous paper on homological properties in relation to Nakayama algebras, we showed that Ext-groups of all pairs (M, N) of modules over Nakayama algebras of type (n, n, n) satisfies the condition $Ext^n_{\Lambda}(M, N) = 0$ for $n \gg 0 \iff Ext^n_{\Lambda}(N, M) = 0$ for $n \gg 0$ using the projectives of lengths $3n + 1$ and $3n + 2$, [1]. The algebra Λ is a Nakayama algebra if every projective indecomposable and every injective indecomposable Λ-module is uniserial. In other words, these modules have a unique composition series, (see Schrer [3]). Nakayama algebras are finite dimensional and representation-finite algebras that have a nice representation theory in the sense that the finite-dimensional indecomposable modules are easy to describe. The main contribution of this paper is to investigate and prove that the Ext-groups of all pairs (M, N) of modules over Nakayama algebras of type (n, n, n) do not satisfy the condition $Ext^n_{\Lambda}(M, N) = 0$ for $n \gg 0 \iff Ext^n_{\Lambda}(N, M) = 0$ for $n \gg 0$.

2 Preliminary

This section will briefly discuss Nakayama algebras and some related propositions. An $R-$algebra Γ is a ring together with ring morphism $\phi : R \longrightarrow \Gamma$ whose image is in the center of Γ. Γ is therefore an Artin algebra if it is finitely generated $R-$module. We define Nakayama algebras in terms of uniserial modules. Let Γ be an Artin algebra. A Γ-module A is called uniserial module if the set of submodules is totally ordered by inclusion. We state the following propositions to verify the properties of Nakayama algebras. All the information presented here can be found in deeper details from [4],[5],[2].

Proposition 2.1. *The following are equivalent for $\Lambda - moduleA$.*

a. *A is uniserial.*

b. *There is only one composition series for A.*

c. *The radical filtration of A is a composition series for A.*

d. *The socle filtration of A is a composition series for A.*

e. *$l(A) = rl(A)$, where $l(A)$ is the length of A and $rl(A)$ is the radical length of A.*

Proof. $a \Rightarrow b$.

A is uniserial implies there is only one composition series for A. Let the following be two composition series for A;

$A = A_0 \supset A_1 \supset A_2 \cdots\cdots \supset A_n$

$A = B_0 \supset B_1 \supset B_2 \cdots\cdots \supset B_n$

Since A is uniserial, $A_1 \subseteq B_1$ or $B_1 \subseteq A_1$. Without loss of generality, we assume $B_1 \subseteq A_1$, then we have the sequence; $0 \longrightarrow A_1/B_1 \longrightarrow A/B_1 \longrightarrow A/A_1 \longrightarrow 0$ such that $A_i/A_{i+1} = 0$ or simple for $i = 0, 1, \cdots, n-1$. Assume that $A_i/A_{i+1} \neq 0$, then A/B_1 is simple, A/A_1 is simple, B_0/B_1 is simple and B_1/B_2 is also simple. A/B_1 is simple because $A \supseteq B_1$ is the start of a composition series hence A/A_1 is also simple. Let the following map be a homomorphism;

$A/B_1 \longrightarrow^{\varphi} A/A_1$. If A/B_1 and A/A_1 are simple modules, then the map is either zero or isomorphism. The $ker\varphi = A_1/B_1$. If $ker\varphi = 0$, then $A_1 = B_1$. By induction on n, $A_2 = B_2$. Hence there is only one composition series for A.

The $rl(A) = l(A)$ and $sl(A) = l(A)$, and therefore, $rl(A) = sl(A)$. Hence, $b \Rightarrow c \Rightarrow d \Rightarrow e$ is trivial.

Let the radical filtration of A be; $A \supset rA \supset r^2A \supset \cdots\cdots \supset r^nA = 0$ with $rl(A) = n$

Let the composition series of A be; $A = A_1 \supset A_2 \supset \cdots\cdots \supset A_n = 0$ which implies $l(A) = n$.

Assume $0 = r^nA \subset r^{n-1}A \subset \cdots\cdots \subset rA \subset A$ is not a composition series but $rl(A) = n$. Consider the sequence; $0 \longrightarrow rA \longrightarrow A \longrightarrow A/rA$ which implies that $l(A) = l(rA) + l(A/rA)$.

Similarly we have the sequence; $0 \longrightarrow r^2A \longrightarrow rA \longrightarrow rA/r^2A \longrightarrow 0$, and hence $l(A) = l(A/rA) + l(r^2A)$. We therefore have $l(A) = \sum l(r^iA/r^{i+1}A) \geq rl(A)$ which is a contradiction. Hence, the $l(A) = rl(A)$, $e \Rightarrow a$. □

Proposition 2.2. *The following are equivalent for an Artin algebra Λ.*

a. *Λ is a sum of uniserial modules .*

b. *Λ/a is a sum of uniserial modules for all ideals a of Λ.*

c. *Λ/r^2 is a sum of uniserial modules.*

Proof. $a \Longrightarrow b$ and $b \Longrightarrow c$ are trivial. If Λ is the sum of uniserial modules, then Λ/a and Λ/r^2 which are factors of Λ are also a sum of uniserial modules.

$c \Longrightarrow a$

Let P be an indecomposable projective $\Lambda-$module. We show that P/r^nP is uniserial by induction on n when $n \geq 2$. When $n = 2$, there is nothing to prove. Suppose $n > 2$. Let the radical filtration of P be; $P \supset rP \supset r^2P \supset \cdots\cdots r^{n-1}P \supset r^nP = 0$ such that $r^iP/r^{i+1}P$ is simple for $i = 0, 1, \cdots, n-1$.

When $n = 3$, we have $r^3P \subset r^2P \subset rP \subset P$. Hence by induction hypothesis, $P/r^{n-1}P$ is uniserial. Considering the exact sequence $0 \longrightarrow rP \longrightarrow P \longrightarrow P/rP \longrightarrow 0$, which also implies that P/rP is uniserial, hence $P/r^{n-1}P$ is also uniserial.

If $r^{n-1}P = 0$, then P/r^nP is clearly uniserial, so we have to assume that $r^{n-1}P \neq 0$. From proposition 1, it follows that $r^iP/r^{i+1}P$ is simple for $i = 0, 1, \cdots, n-2$. To show that P/r^nP is uniserial, then it is sufficient by proposition 1 to prove that $r^{n-1}P/r^nP$ is also simple.

Let $Q \longrightarrow r^{n-2}P$ be a projective cover. Since $r^{n-2}P/r^{n-1P}$ is simple, Q must be indecomposable and so Q/r^2Q is uniserial. But we have an epimorphism $rQ/r^2Q \longrightarrow r^{n-1}P/r^nP$ which shows that $r^{n-1}P/r^nP$ is simple. □

Proposition 2.3. *Let φ be a D Tr-orbit of indΛ. Suppose there is a projective module P in φ. Then we have the following;*

1. *φ of non-zero objects in $\{P, (DT_r)^{-1}P, \cdots, (DT_r)^{-i}P, \cdots\}_{i\in N}$.*

2. *φ is finite if and only if $(DT_r)^{-n}P = (T_rD)^n$ is injective for some n in N. Moreover, if $(T_rD)^nP$ is injective, then $\varphi = \{P, (DT_r)^{-1}P, \cdots, (DT_r)^{-n}P\}$.*

Proof. The statement, $DTrP = 0$ if and only if P is projective is trivial. Since P is projective module in $\varphi, (DTr)^iP = 0$ for all $i > 0$. Hence the claim in 2(a). We claim that if $(DTr)^{-i}P \simeq (TDr)^{-(i+j)}P \neq 0$ with $j > 0$ we have $(DTr)^i(DTr)^{-(i+j)}P \simeq (DTr)(DTr)^{-i}P$ which implies that $P \simeq (DTr)^jP = (TrD)^jP$ which is not possible since $j > 0$. φ can therefore be finite if $(DTr)^{-(n+1)}P = 0$ for some $n \geq 0$.

Since $(DTr)^{-n}P = (TrD)^nP$, then P is injective in φ. We know therefore that if $(DTr)^{-n}P$ is injective, then $\varphi = \{P, (DTr)^{-1}P, \cdots, (DTr)^{-n}P\}$. □

Proposition 2.4. *Suppose φ contains as injective module I. Then we have the following;*

i. *φ consists of the nonzero modules in $\{I, DTrI, \cdots, (DTr)^iI, \cdots\}_{i\in N}$.*

ii. *φ is finite if and only if $(DTr)^nI$ is projective for some $n \in N$. Moreover, if $(DTr)^n$ is projective, then $\varphi = \{I, (DTr)I, \cdots, (DTr)^nI\}$.*

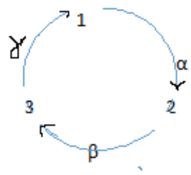
Proof. We know that $TrDP = 0$ if and only if P is injective. So since I is an injective module in φ, $(TrD)^iI = 0$ for all $i > 0$. Hence, the claim in (b).

We claim that $(DT_r)^iI \simeq (DTr)^{(i+j)}I \neq 0$ with $j > 0$. By this claim, we have $(DTr)^{-i}(DTr)^iI \simeq (DT^r)^{-i}(DTr)^{(i+j)}I$ which implies that $I \simeq (DTr)^jI$ for I injective. If $(DTr)^nI$ is projective, then we claim $\varphi = \{I, (DTr)I, \cdots, (DTr)^nI\}$. φ can therefore be finite if $(DTr)^{n+1}I = 0$ for some $n \geq 0$. □

3 Main Work

In this section we discuss examples of the Nakayam algebra with projectives of length $3n$ and $4n$ which do not satisfy the condition: $Ext^n_\Lambda(M,N)=0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N,M)=0$ for $n \gg 0$.

We discuss the Ext-groups of Nakayama algebra with projectives of length $3n$. Let $\Gamma =$



with relations $\gamma\beta\alpha\cdots\beta\alpha, \alpha\gamma\beta\cdots\gamma\beta$ and $\beta\alpha\gamma\cdots\alpha\gamma\beta$, where the length of each relation is $3n$. Let $\Lambda = k\Gamma/\langle\gamma\beta\alpha\cdots\beta\alpha, \alpha\gamma\beta\cdots\gamma\beta, \beta\alpha\gamma\cdots\alpha\gamma\beta\rangle$. The projectives of the above path algebra are as follows;

$$P_1 = (S_1, S_2, \cdots, S_3)^t,\ P_2 = (S_2, S_3, \cdots, S_1)^t,\ P_3 = (S_3, S_1, \cdots, S_2)^t.$$

The above projectives P_1, P_2 and P_3 have length of $3n$ each. The minimal projective resolution of the module S_1 is given as;

$$\cdots \to Q_4 \overset{d_4}{\to} Q_3 \overset{d_3}{\to} Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{d_0}{\to} S_1 \to 0$$

where $Q_{2i} = Q_{2i+2} = P_1$ and $Q_{2i+1} = Q_{2i+3} = P_2$ for $i \geq 0$ and d_{2i+1} is a multiplication by α and d_{2i+2} is a multiplication by $\gamma\beta(\alpha\gamma\beta)^{n-1}$. From the above resolution we have

$$\cdots \to \Lambda e_1 \overset{d_3}{\to} \Lambda e_2 \overset{d_2}{\to} \Lambda e_1 \overset{d_1}{\to} S_1 \overset{d_0}{\to} 0.$$

The $pdS_1 = \infty$ since the resolution is periodic, where pd is the projective dimension. The period is 2. The truncation of the above resolution is given as;

$$P_{S_1} \cdots \Lambda e_1 \overset{d_4}{\to} \Lambda e_2 \overset{d_3}{\to} \Lambda e_1 \overset{d_2}{\to} \Lambda e_2 \overset{d_1}{\to} \Lambda e_1 \overset{d_0}{\to} 0.$$

Applying $Hom_\Lambda(\ ,M)$ where M is any module, we have

$$\begin{aligned} 0 &\overset{d_0^\times}{\to} Hom_\Lambda(\Lambda e_1, M) \overset{d_1^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_2^\times}{\to} Hom_\Lambda(\Lambda e_1, M) \\ &\overset{d_3^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_4^\times}{\to} Hom_\Lambda(\Lambda e_1, M) \end{aligned}$$

where $Hom_\Lambda(\Lambda e_1, M) \simeq e_1M$ and $Hom_\Lambda(\Lambda e_2, M) \simeq e_2M$. By definition, we have $Ext^i_\Lambda(S_1, M) = ker(d^\times_{i+1})/\Im(d^\times_i)$. We calculate the Ext-groups.

$$\begin{aligned} Ext^1_\Lambda(S_1, M) &= ker\left(e_2M \xrightarrow{\gamma\beta(\alpha\gamma\beta)^{n-1}} e_1M\right)/\Im\left(e_1M \overset{\alpha}{\to} e_2M\right) \\ &= \left\{e_2m|\gamma\beta(\alpha\gamma\beta)^{n-1}e_2m = 0\right\}/\alpha. \end{aligned}$$

Let $M = S_2$, we have

$$Ext^1_\Lambda(S_1, S_2) = \left\{e_2S_2|\gamma\beta(\alpha\gamma\beta)^{n-1}e_2S_2 = 0\right\}/\alpha e_1S_1 = e_2\dot{S}_2/\alpha e_1S_2 = k/0 = k.$$

This implies that for $Ext^i_\Lambda(S_1, S_2)$ not all values are zero for $i \gg 0$. The minimal projective resolution of the module S_2 is as follows;

$$\cdots \to Q_4 \overset{d_4}{\to} Q_3 \overset{d_3}{\to} Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{d_0}{\to} S_2 \to 0$$

where $Q_{2i} = Q_{2i+2} = P_2$ and $Q_{2i+1} = Q_{2i+3} = P_3$ for $i \geq 0$ and d_{2i+1} is a multiplication by β and d_{2i+2} is a multiplication by $\alpha\gamma(\beta\alpha\gamma)^{n-1}$. The projectives P_2 and P_3 all have length $3n$ each. From the resolution we have the following;

$$\cdots \to \Lambda e_2 \overset{d_4}{\to} \Lambda e_3 \overset{d_3}{\to} \Lambda e_2 \overset{d_2}{\to} \Lambda e_3 \overset{d_1}{\to} \Lambda e_2 \overset{d_0}{\to} S_2 \to 0.$$

The $pdS_2 = \infty$ since the resolution is periodic. The period is 2. The truncation of the resolution is given below,

$$P_{S_2} \cdots \Lambda e_2 \overset{d_4}{\to} \Lambda e_3 \overset{d_3}{\to} \Lambda e_2 \overset{d_2}{\to} \Lambda e_3 \overset{d_1}{\to} \Lambda e_2 \overset{d_0}{\to} 0.$$

Applying $Hom_\Lambda(\ ,M)$ where M is any given module, we have

$$0 \overset{d_0^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_1^\times}{\to} Hom_\Lambda(\Lambda e_3, M) \overset{d_2^\times}{\to} Hom_\Lambda(\Lambda e_2, M)$$
$$\overset{d_3^\times}{\to} Hom_\Lambda(\Lambda e_3, M) \overset{d_4^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \to 0$$

where $Hom_\Lambda(\Lambda e_2, M) \simeq e_2 M$ and $Hom_\Lambda(\Lambda e_3, M) \simeq e_3 M$.

We calculate the Ext-groups.

$$Ext^1_\Lambda(S_2, M) = ker\left(e_3 M \xrightarrow{\alpha\gamma(\beta\alpha\gamma)^{n-1}} e_2 M\right)/\Im\left(e_2 M \overset{\beta}{\to} e_3 M\right)$$
$$= \left\{e_3 m | \alpha\gamma(\beta\alpha\gamma)^{n-1} e_3 m = 0\right\}/\beta.$$

Let $M = S_1$, then we have

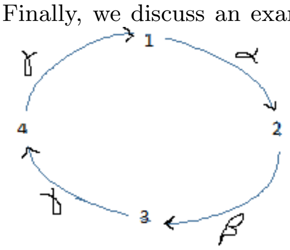
$$Ext^1_\Lambda(S_2, S_1) = \left\{e_3 S_1 | \alpha\gamma(\beta\alpha\gamma)^{n-1} e_3 S_1 = 0\right\}/\beta e_2 S_1 = e_3 \dot{S}_1/\beta e_2 S_1 = 0.$$

$$Ext^2_\Lambda(S_2, S_1) = ker\left(e_2 S_1 \overset{\beta}{\to} e_3 S_1\right)/\Im\left(e_3 S_1 \xrightarrow{\alpha\gamma(\beta\alpha\gamma)^{n-1}} e_2 S_1\right)$$
$$= \{e_2 S_1 | \beta e_2 S_1 = 0\}/\alpha\gamma(\beta\alpha\gamma)^{n-1} = e_2 S_1/\alpha\gamma(\beta\alpha\gamma)^{n-1} e_3 S_1 = 0.$$

The above Ext-groups imply that for $Ext^i_\Lambda(S_2, S_1)$, all values are zero for $i \gg 0$.

We conclude that the Ext-groups of the Nakayama algebra with projectives of length $3n$ do not satisfy the condition $Ext^n_\Lambda(M, N) = 0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N, M) = 0$ for $n \gg 0$

Finally, we discuss an example of the Nakayama algebra with projectives of length $4n$. Let $\Gamma =$



with relations $\delta\gamma\beta\alpha \cdots \delta\gamma\beta\alpha, \alpha\delta\gamma\beta \cdots \alpha\delta\gamma\beta, \beta\alpha\delta\gamma \cdots \beta\alpha\delta\gamma$ and $\gamma\beta\alpha\delta \cdots \gamma\beta\alpha\delta$, where the length of each relation is $4n$, n is a positive integer. Let $\Lambda = k\Gamma/\langle \delta\gamma\beta\alpha \cdots \delta\gamma\beta\alpha, \alpha\delta\gamma\beta \cdots \alpha\delta\gamma\beta, \beta\alpha\delta\gamma \cdots \beta\alpha\delta\gamma, \gamma\beta\alpha\delta \cdots \gamma\beta\alpha\delta\rangle$. Let the projectives of the above path be;

$$P_1 = (S_1, S_2, \cdots, S_3, S_4)^t,\ P_2 = (S_2, S_3, \cdots, S_4, S_1)^t,$$
$$P_3 = (S_3, S_4, \cdots, S_1, S_2)^t,\ P_4 = (S_4, S_1, \cdots, S_2, S_3)^t.$$

The above projectives P_1, P_2, P_3 and P_4 have length $4n$ each. The minimal projective resolution of the module S_1 is given as;

$$\cdots \to Q_4 \overset{d_4}{\to} Q_3 \overset{d_3}{\to} Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{d_0}{\to} S_1 \to 0$$

where $Q_{2i} = Q_{2i+2} = P_1$ and $Q_{2i+1} = Q_{2i+3} = P_2$ for $i \geq 0$ and d_{2i+1} is a multiplication by α and d_{2i+2} is a multiplication by $\delta\gamma\beta(\alpha\delta\gamma\beta)^{n-1}$. From the above resolution we have;

$$\cdots \to \Lambda e_1 \overset{d_4}{\to} \Lambda e_2 \overset{d_3}{\to} \Lambda e_1 \overset{d_2}{\to} \Lambda e_2 \overset{d_1}{\to} \Lambda e_1 \overset{d_0}{\to} S_1 \to 0.$$

The $pdS_1 = \infty$ since the resolution is periodic. The period is 2. The truncation of the resolution is given as;

$$P_{S_1} \cdots \to \Lambda e_1 \overset{d_4}{\to} \Lambda e_2 \overset{d_3}{\to} \Lambda e_1 \overset{d_2}{\to} \Lambda e_2 \overset{d_1}{\to} \Lambda e_1 \overset{d_0}{\to} 0.$$

Applying $Hom_\Lambda(\ , M)$, where M is any given module, we have

$$0 \overset{d_0^\times}{\to} Hom_\Lambda(\Lambda e_1, M) \overset{d_1^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_2^\times}{\to} Hom_\Lambda(\Lambda e_1, M)$$
$$\overset{d_3^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_4^\times}{\to} Hom_\Lambda(\Lambda e_1, M)$$

where $Hom_\Lambda(\Lambda e_1, M) \simeq e_1 M$ and $Hom_\Lambda(\Lambda e_2, M) \simeq e_2 M$.

We calculate the Ext-groups.

$$Ext^1_\Lambda(S_1, M) = ker\left(e_2 M \xrightarrow{\delta\gamma\beta(\alpha\delta\gamma\beta)^{n-1}} e_1 M\right) / \Im\left(e_1 M \overset{\alpha}{\to} e_2 M\right)$$
$$= \left\{e_2 m | \delta\gamma\beta(\alpha\delta\gamma\beta)^{n-1} e_2 m = 0\right\} / \alpha e_1 M.$$

Let $M = S_2$, then we have

$$Ext^1_\Lambda(S_1, S_2) = \left\{e_2 S_2 | \delta\gamma\beta(\alpha\delta\gamma\beta)^{n-1} e_2 S_2 = 0\right\} / \alpha e_1 S_2 = e_2 \dot{S}_2 / \alpha e_1 S_2 = k.$$

This implies that for $Ext^i_\Lambda(S_1, S_2)$, not all values are zero for $i \gg 0$. The minimal projective resolution of the module S_2 is as follows;

$$\cdots \to Q_4 \overset{d_4}{\to} Q_3 \overset{d_3}{\to} Q_2 \overset{d_2}{\to} Q_1 \overset{d_1}{\to} Q_0 \overset{d_0}{\to} S_2 \to 0$$

where $Q_{2i} = Q_{2i+2} = P_2$ and $Q_{2i+1} = Q_{2i+3} = P_3$ and d_{2i+1} is a multiplication by β and d_{2i+2} is a multiplication by $\alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1}$. From the resolution, we have

$$\cdots \to \Lambda e_2 \overset{d_4}{\to} \Lambda e_3 \overset{d_3}{\to} \Lambda e_2 \overset{d_2}{\to} \Lambda e_1 \overset{d_1}{\to} \Lambda e_2 \overset{d_0}{\to} S_2 \to 0.$$

The $pdS_2 = \infty$ since the resolution is periodic and the period is 2. The truncation of the resolution is given below;

$$P_{S_2} \cdots \Lambda e_2 \overset{d_4}{\to} \Lambda e_3 \overset{d_3}{\to} \Lambda e_2 \overset{d_2}{\to} \Lambda e_3 \overset{d_1}{\to} \Lambda e_2 \overset{d_0}{\to} 0.$$

Applying $Hom_\Lambda(\ , M)$, we have

$$0 \overset{d_0^\times}{\to} Hom_\Lambda(\Lambda e_2, M) \overset{d_1^\times}{\to} Hom_\Lambda(\Lambda e_3, M) \overset{d_2^\times}{\to} Hom_\Lambda(\Lambda e_2, M)$$
$$\overset{d_3^\times}{\to} Hom_\Lambda(\Lambda e_3, M) \overset{d_4^\times}{\to} Hom_\Lambda(\Lambda e_2, M)$$

where $Hom_\Lambda(\Lambda e_2, M) \simeq e_2 M$ and $Hom_\Lambda(\Lambda e_3, M) \simeq e_3 M$.

We calculate the Ext-groups.

$$Ext^1_\Lambda(S_2, M) = ker\left(e_3 M \xrightarrow{\alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1}} e_2 M\right) / \Im\left(e_2 M \overset{\beta}{\to} e_3 M\right)$$
$$= \left\{e_3 m | \alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1} e_3 m = 0\right\}.$$

Let $M = S_1$, we have

$$Ext^1_\Lambda(S_2, S_1) = \left\{e_3S_1|\alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1}e_3S_1 = 0\right\}/\beta e_2S_1 = e_3S_1/\beta e_2S_1 = 0.$$

$$\begin{aligned} Ext^2_\Lambda(S_2, S_1) &= ker\left(e_2M \overset{\beta}{\to} e_3M\right)/\Im\left(e_3M \xrightarrow{\alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1}} e_2M\right) \\ &= \{e_2S_1|\beta e_2S_1 = 0\}/\alpha\delta\gamma(\beta\alpha\delta\gamma) = e_2S_1/\alpha\delta\gamma(\beta\alpha\delta\gamma)^{n-1}e_3S_1 = 0. \end{aligned}$$

The above Ext-groups show that for $Ext^i_\Lambda(S_2, S_1)$, all values are zero for $i \gg 0$. We therefore conclude that the condition $Ext^n_\Lambda(M, N) = 0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N, M) = 0$ for $n \gg 0$ does not hold for Nakayama algebra with projectives of length $4n$.

4 Conclusion

We conclude that the condition $Ext^n_\Lambda(M, N) = 0$ for $n \gg 0 \Longleftrightarrow Ext^n_\Lambda(N, M) = 0$ for $n \gg 0$ does not hold for Ext-groups of Nakayama algebra with projectives of lengths $3n$ and $4n$.

Competing Interests

Authors have declared that no competing interests exist.

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