Journal of Advances in Mathematics and Computer Science

28(2): 1-18, 2018; Article no.JAMCS.42407

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

Solving Different Order Boundary Value Problems by Applying Homotopy Type Methods

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Authors' contributions

This work was carried out in equal collaboration of all the authors. The final version of the manuscript read and approved by the all authors.

Article Information

DOI: 10.9734/JAMCS/2018/42407
Editor(s):
(1) Carlo Bianca, Professor, Laboratoire de Physique Théorique de la Matière Condensée, Sorbonne Universités, France.
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(3) Omar Abu Arqub, Al-Balqa Applied University, Jordan.
Complete Peer review History: http://www.sciencedomain.org/review-history/25650

Original Research Article

Received: 10th May 2018
Accepted: 17th July 2018
Published: 23rd July 2018

Abstract

In this manuscript, the methods, Optimal Homotopy Asymptotic Method (OHAM), Homotopy Perturbation Method (HPM) and Homotopy Analysis Method (HAM) are applied to solve various order boundary value problems. These techniques give solutions in the form of a series. Their solutions and graphs are compared with their exact solutions and graphs. All these give profitable results, but the Optimal Homotopy Asymptotic Method is more precise, whose convergence is restrained optimizations and the convergence area can be accustomed affording to the problem concerned. The results show that the suggested scheme is more active and relaxed to routine.

Keywords: OHAM; HPM; HAM; various order boundary value problems.

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1 Introduction

Differential equations can be used to model various types of physical structures such as sociological, economical, biological and chemical, etc. Also in literature, physical problems are investigated by differential equations, which are mostly handled by the common methods, Splines (S) [1], Homotopy perturbation method (HPM) [2], Homotopy Analysis Method (HAM) [3,4]. The non-perturbed techniques (DTM) [5], and ADM) [6,7] concern nonlinear problems, but the region of convergence of their series solution is generally small. Freshly Herisanu and Marinca et al. [8-10] familiarized OHAM for approximate solution of nonlinear problems of shrill film flow of a fourth grade fluid down an upright cylinder. They used the whispered method for understanding the conduct of nonlinear mechanical vibration of an electric contraption. By using this method they investigated the solution of nonlinear equations ascending in the study of state-run flow of a fourth grade fluid past a porous plate. This method supplies the need to switch the convergence. In general the OHAM solution settles with the exact solution. The graphs of the two solutions are conterminous. We have smeared OHAM to various types of boundary value problems and have investigated that the original exact solution approves with the numerical solution, the error noted is small. This method is operational and is relaxed to routine. The other relevant work [5,11-16] has been observed in these articles. Moreover the analytical solution of various differential equations are discussed in [17,18, 19,20,21,22,23,24,25,26,27,28,29,30] and [31-34] in detail.

2 Analysis of the Method OHAM

Considering the boundary problem of the arrangement

$$L(F(y))+g(y)+N(F(y))=0,\ B\left(F,\frac{dF}{dy}\right)=0 \tag{2.0 a}$$

Where L is taken as linear operator y is independent variable $F(y)$ is an unknown function $g(y)$ is a notorious function, $N(F(y))$ is a nonlinear operator and B is a boundary operator. According to the idea of OHAM we construct a homotopy as given below

$$H(\theta(y,p),p):R\times[0,1]\rightarrow R$$

That satisfies the homotopy functional as given below

$$(1-p)[L(\theta(y,p))+g(y)]=h(p)[L(\theta(y,p))+g(y)+N(\theta(y,p))],\ B\left(\theta(y,p),\frac{\partial(\theta(y,p)}{\partial y}\right)=0, \tag{2.0 b}$$

Where, $y\in R$, $p\in[0,1]$ is an embedding parameter, $h(p)$ is a nonzero auxiliary function for $p\neq 0$, $h(0)=0$ and $\theta(y,p)$ is an Unknown function .Evidently, for p=0 and p=1.Manifestly, for p=0 and p=1 it restrains that $\theta(y,1)=F(y)$ respectively. Thus as p varies from 0 to 1 , the solution $\theta(y,p)$ approaches from $F_0(y)$

to $F_1(y)$ where $F_0(y)$ is obtained from Equation (2.0 b) for $p=0$ and we have.

$$L(F_0(y))+g(y)=0,\ B\left(F_0,\frac{dF_0}{dy}\right)=0, \tag{2.0 c}$$

Now choosing auxiliary function $h(p)$ in the following pattern

$$h(p) = pC_1 + p^2C_2 + ..., \tag{2.0 d}$$

Where $C_1, C_2 \cdots$ are constants to be determined later. $h(p)$ can be uttered in many customs as investigated by V. Marinca [8-10]. To acquire an approximate solution one can expand $\theta(y, p, C_i)$ in Taylor's series about p in the following Pattern.

$$\theta(y, p, C_i) = F_0(y) + \sum_{k=1}^{\infty} F_k(y, C_1, C_2, ...C_k) p^k, \tag{2.0 e}$$

Making use of equation (2.0 e) in equation (2.0 b) and comparing the coefficients of resembling powers of p we have the following linear equations zeroth order problem is given by equation (2.0 c) and the first and second order problems are given by equations (2.0 f) and (2.0 g)

$$L(F_1(y)) + g(y) = C_1 N_0(F_0(y)), B\left(F_1, \frac{dF_1}{dy}\right) = 0, \tag{2.0 f}$$

$$L(F_2(y)) - L(F_1(y)) = C_2 N_0(F_0(y)) + C_1[L(F_1(y)) + N_1(F_0(y), F_1(y))], B\left(F_2, \frac{dF_2}{dy}\right) = 0, \tag{2.0 g}$$

The general governing equations for $F_k(y)$ are given by

$$\begin{aligned} &L(F_k(y)) - L(F_{k-1}(y)) = C_k N_0(F_0(y)) + \\ &\sum_{i=1}^{k-1} C_i[L(F_{k-i}(y)) + N_{k-i}(F_0(y), F_0(y), ..., F_{k-1}(y)), k = 2,3..., B\left(F_k, \frac{dF_k}{dy}\right) = 0, \end{aligned} \tag{2.0 h}$$

Where $N_m(F_0(y), F_1(y)..., F_{k-1}(y))$ is the coefficient of p^m in the expansion of $N(\theta(y, p))$ about the embedding Parameter p

$$N(\theta(y, p, C_i)) = N_0(F_0(y)) + \sum_{m=1}^{\infty} N_m(F_0, F_1...F_m) p^m, \tag{2.0 i}$$

It has been investigated that the convergence of the series (2.0 e) depends on the auxiliary constants C_1, C_2,.. If it is convergent. At $p = 1$ then we have

$$\theta(y, C_i) = F_0(y) + \sum_{k=1}^{\infty} F_k(y, C_1, C_2...C_k), \tag{2.0 j}$$

The upshot of the mth order approximations are

$$\tilde{F}(y, C_1, C_2, ...C_m) = F_0(y) + \sum_{i=1}^{m} F_i(y, C_1, C_2, ...C_k), \tag{2.0 k}$$

Using equation (2.0 k) into equation (2.0 a) we get the residual

$$R(y, C_1, C_2 ..., C_m) = L(\tilde{F}(y, C_1, C_2 ..., C_m)) + g(y) + N(\tilde{F}(y, C_1, C_2 ..., C_m)), \quad (2.0\ l)$$

If $R = 0$ then $\tilde{F}$ be the exact solution. Generally it does not happen especially in nonlinear problems .In order to get the Optimal Values of C_i's i=1,2,3… we first construct the Functional Values of C_i's i=1,2,3… we first construct the Functional.

$$J(C_1, C_2, ..., C_m) = \int_a^b R^2(y, C_1, C_2, ..., C_m) dy, \quad (2.0\ m)$$

And then minimizing it using the basic calculus we get

$$\frac{\partial J}{\partial C_1} = 0, \frac{\partial J}{\partial C_2} = 0, ..., \frac{\partial J}{\partial C_m} = 0. \quad (2.0\ n)$$

Knowing the values of $C_1, C_2 \cdots C_m$. The approximate solution of order m is determined. Where a, b lie in domain of the concerned problem using the least square method we get OHAM solution.

2.1 Fundamentals of HPM

We show the basic idea of the homotopy perturbation method [2] and considering the nonlinear differential equation of the form:

$$C(z) - g(s) = 0, s \in \Omega \quad (2.1\ a)$$

With boundary conditions

$$G\left(z, \frac{dz}{dn}\right) = 0, s \in \Gamma \quad (2.1\ b)$$

Where C is a general differential operator, G is a boundary operator, z is a known analytical function, Γ is the boundary of the domain Ω. The operator C can be divided into two parts, E and W are linear and nonlinear. Equation (2.1a) takes the form:

$$E(z) + W(z) - g(s) = 0, \quad (2.1\ c)$$

By the homotopy method proposed by Liao [4]. A homotopy can be constructed as

$v(s, j) : \Omega \times [0,1] \to R$ This satisfies:

$$T(v, j) = (1 - j)[E(v) - E(z_0)] + j[C(v) - g(s)] = 0, \quad (2.1\ d)$$

or $T(v, j) = E(v) - E(z_0) + jE(z_0) + j[W(v) - g(s)] = 0,$ (2.1 e)

Where $s \in \Gamma$ and $j \in [0,1]$ is an embedding parameter, z_0 is an initial approximation of [3] which satisfies the boundary conditions using (2.1 d) we can easily guess that

$$T(v,0) = E(v) - E(z_0) = 0, \tag{2.1 f}$$

$$T(v,1) = C(v) - g(s) = 0, \tag{2.1 g}$$

And the altering technique of j from zero to one is just that of $T(v, j)$ from $E(v) - E(z_0)$ to $C(v) - g(s)$. In the Topology, this is called deformation, where $C(v) - g(s)$ and $E(v) - E(z_0)$ are called homotopic. The embedding parameter j is known initially. For $0 < j \leq 1$ equation (2.1 d) can be given as

$$v = v_0 + jv_1 + j^2 v_2 + \cdots, \tag{2.1 h}$$

The approximate solution of equation of (18) can be obtained as follows:

$$z = \lim_{j \to 1} v = v_0 + v_1 + v_2 + \cdots, \tag{2.1 i}$$

2.2 Fundamentals of HAM

We start by considering

$$N[u(x)] = 0, \tag{2.2 a}$$

Where N a nonlinear operator is $u(x)$ is an unknown function and x denotes independent variable. For simplicity we ignore all boundary conditions. We construct zeroth order functional as:

$$(1-q)L[\phi(x, q) - u_0(x)] = qhH(x)N[\phi(x,q)], \tag{2.2 b}$$

Where $q \in [0,1]$ is an embedding parameter, $h \neq 0$ is a nonzero auxiliary parameter $H(x)$ is an auxiliary function L

Is a linear operator $u_0(x)$ is an initial guess of $u(x)$, $\phi(x,q)$ is an unknown function , from equation (2.2b) $q = 0$ and $q = 1$ we get

$$\phi(x,0) \text{ and } \phi(x,1), \tag{2.2 c}$$

Thus as q increases from 0 to 1, the solution $\phi(x,q)$ varies from the $u_0(x)$ to $u(x)$. Expanding $\phi(x,q)$ in Taylor series

We have

$$\phi(x,q) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x) q^m, \tag{2.2 d}$$

Where

$$u_m(x) = \frac{1}{m!}\frac{\partial^m \phi(x,q)}{\partial q^m}, \text{ at } q = 0 \quad (2.2\text{ e})$$

If the linear operator, initial guess, the auxiliary parameter and auxiliary function are so properly chosen the series (2.3 e) Converges at $q = 1$ We have

$$u(x) = u_0(x) + \sum_{m=1}^{+\infty} u_m(x), \quad (2.2\text{ f})$$

Which is one of the solution of the original non-linear equation approved by Liao [3] choosing $h = -1$ and $H(x) = 1$

Equation (2.2 b) becomes

$$(1-q)L[\phi(x,q) - u_0(x)] + qN[\phi(x,q)] = 0, \quad (2.2\text{ g})$$

Which is used mostly in Homotopy Perturbation Method (HPM) whereby the solution is obtained directly, without using Taylor series the comparison between HPM and HAM can be found in [3]. As $H(x) = 1$ equation (2.2 b) becomes

$$(1-q)L[\phi(x,\ q) - u_0(x)] = qhN[\phi(x,q)], \quad (2.2\text{ h})$$

This is often used in (HAM). In this case $H(x)$ will not be involved in setting of the base function. Differentiating equation (2.2b) m times with respect to q and the q=0 and finally dividing them by $m!$ we have the so called m^{th}-order deformation equation

$$L[u_m(x) - \chi_m u_{m-1}(x)] = hH(x)R_m(\overline{u}_{m-1,}\,x), \quad (2.2\text{ i})$$

$$R_m(\overline{u}_{m-1}, x) = \frac{1}{(m-1)!}\frac{\partial^{m-1} N[\phi(x,q)}{\partial q^{m-1}}, \text{ at } q = 0 \quad (2.2\text{ j})$$

$$\overline{u}_m = \{u_0(x), u_1(x), \cdots, u_m(x)\}, \quad (2.2\text{ k})$$

And $\chi = \{0, m \le 1, \quad \chi = \{1, m > 1$ putting equation (2.2 d) in (2.2 j) we have

$$R_m(\overline{u}_{m-1}, x) = \frac{1}{(m-1)!}\frac{\partial^{m-1}}{\partial q^{m-1}} N\left[\sum_{n=0}^{+\infty} u_n(x) q^n\right]\text{. at } q = 0 \quad (2.2\text{l})$$

It should be restrained that $u_m(x)$ for $m \ge 1$ is governed by the linear equation (2.2 i) with the linear boundary condition that is obtained from the original problem which can be solved by Matlab.

3 Numerical Problems

Problem 3.1.

Consider the following linear differential equation

$$F'(y)-F(y)-y\cos y+y\sin y-\sin y=0\,,F(0)=0, \tag{3.1 a}$$

The exact solution of the problem is

$$F(y)=y\sin y, \tag{3.1 b}$$

Applying the method mentioned above, the zeroth order problem is

$$F_0'(y)=0\,,F_0(0)=0, \tag{3.1 c}$$

Its solution is

$$F_0(y)=0, \tag{3.1 d}$$

First order problem is

$$F_1'(y,C_1)=-y\cos y\,C_1-\sin y\,C_1+y\sin y\,C_1-C_1F_0(y)+(1+C_1)F_0'(y),F_1(0)=0, \tag{3.1 e}$$

Its solution is

$$F_1(y,C_1)=-y\cos y\,C_1+\sin y\,C_1-y\sin y\,C_1, \tag{3.1 f}$$

Second order problem is

$$F_2'(y,C_1,C_2)=\begin{pmatrix}-y\cos y\,C_2-\sin y\,C_2+y\sin y\,C_2-C_2F_0(y)-C_1F_1(y)+C_2F_0'(y)+(1+C_1)F_1',\\ F_2(0)=0\end{pmatrix}, \tag{3.1 g}$$

Its solution is

$$\begin{aligned}F_2(y,C_1,C_2)=&-y\cos y\,C_1+\sin y\,C_1-y\sin y\,C_1-2C_1^2+2\cos y\,C_1^2-2y\cos y\,C_1^2+2\sin y\,C_1^2\\&-y\cos y\,C_2+\sin y\,C_2-y\sin y\,C_2,\end{aligned} \tag{3.1 h}$$

Third order problem is

$$\begin{aligned}&F_3'(y,C_1,C_2,C_3)=-y\cos y\,C_3-\sin y\,C_3+y\sin y\,C_3-C_3F_0(y)-C_2F_1(y)-C_1F_2(y)\\&+C_3F_0'(y)+C_2F_1'(y)+(1+C_1)F_2'(y),F_3(0),\end{aligned} \tag{3.1 i}$$

Its solution is

$$\begin{aligned}&F_3(y,C_1,C_2,C_3)=\\&-y\cos y\,C_1+\sin y\,C_1-y\sin y\,C_1-4C_1^2+4\cos y\,C_1^2-4y\cos y\,C_1^2+4\sin y\,C_1^2-6\,C_1^3\\&+2y\,C_1^3+6\cos y\,C_1^3-2y\cos y\,C_1^3+2y\sin y\,C_1^3-y\cos y\,C_2+\sin y\,C_2-y\sin y\,C_2\\&-4C_1C_2+4\cos y\,C_1C_2-4y\cos y\,C_1C_2+4\sin y\,C_1C_2-y\cos y\,C_3+\sin y\,C_3-y\sin y\,C_3,\end{aligned} \tag{3.1 j}$$

Now we use equations (3.1 d), (3.1 f), (3.1 h), (3.1 j), and the third order approximate solution by OHAM for p=1 is

$$\tilde{F}(y,C_1,C_2,C_3)=F_0(y)+F_1(y,C_1)+F_2(y,C_1,C_2)+F_3(y,C_1,C_2,C_3). \tag{3.1 k}$$

Using the technique mentioned in section 2 on the domain $a=0, b=1$.we use the residual

$$R=\tilde{F}'-\tilde{F}-y\cos y+y\sin y-\sin y \tag{3.1 l}$$

The following values of C_1, C_2, C_3 are obtained

$$C_1=-1.1763684395430054, C_2=0.11688800369849017, C_3=-0.0026970142170896472, \tag{3.1 m}$$

Considering the values of C_1, C_2, C_3, the approximate OHAM solution becomes

$$\begin{aligned}\tilde{F}(y)=&\\ &1.01231y^2-0.0480322y^3-0.105409y^4-0.0223286y^5+0.00421533y^6+0.00112045y^7\\ &-0.0000878039y^8-0.0000237398y^9+1.11482\times10^{-6}y^{10}+2.90161\times10^{-7}y^{11}\\ &-9.50035\times10^{-9}y^{12}-2.33658\times10^{-9}y^{13}+5.79949\times10^{-11}y^{14}+1.3396\times10^{-11}y^{15}+O(y^{16}),\end{aligned} \tag{3.1 n}$$

The HPM solution is

$$\begin{aligned}&\tilde{F}(y)=\\ &y^2-0.333333y^3-0.166667y^4+0.033333y^5+y0.008333y^6-0.001190y^7-0.000198y^8+O(y^9),\end{aligned} \tag{3.1 o}$$

The HAM solution is

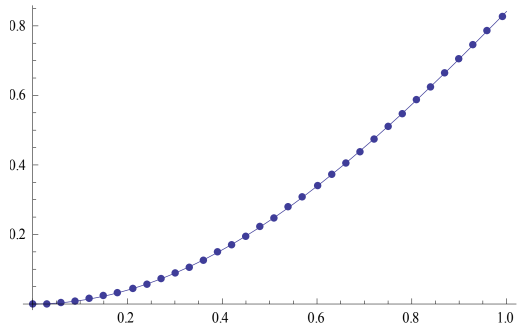
$$\tilde{F}(y)=1.0026y^2-0.021482y^3-0.12265y^4-0.0223774y^5+0.00539169y^6+0.00109117y^7+O(y^8). \tag{3.1 p}$$

The following Table 3.1 displays values of exact solution (3.1 b) OHAM solution (3.1 n) and the error.

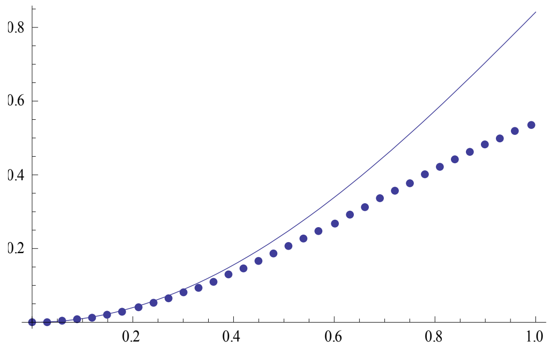
In the Table 3.1 the values of HPM, HAM and their errors are also given. From the table given above we conclude that the errors of the technique, OHAM are smaller than the errors of HPM and HAM.

Table 3.1.

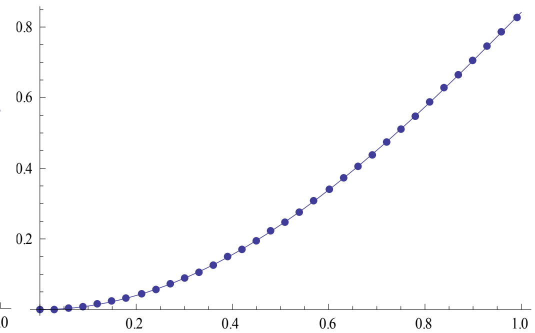
y	**Exact sol**	**OHAM sol**	**HPM sol**	**HAM sol**	**Er OHAM**	**Er HPM**	**Er HAM**
0.0	0.000000	0.000000	0.000000	0.0000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	0.00998334	0.0100643	0.00965034	0.00999199	8.0 E-6	3.3 E-4	8.0 E-6
0.2	0.0397379	0.0399325	0.0370779	0.0397289	1.9 E-4	2.6 E-3	4.9 E-6
0.3	0.0886561	0.0889059	0.0797368	0.0886099	2.4 E-4	8.9 E-3	4.6 E-5
0.4	0.155767	0.155987	0.134773	0.155695	2.1 E-4	2.0 E-2	7.2 E-5
0.5	0.239713	0.239861	0.199079	0.239691	1.4 E-4	4.0 E-2	2.1 E-5
0.6	0.33785	0.333884	0.269344	0.338939	9.8 E-5	6.9 E-2	1.5 E-4
0.7	0.450952	0.451076	0.342124	0.451411	1.2 E-4	1.0 E-1	4.5 E-4
0.8	0.573885	0.574113	0.413894	0.574712	2.2 E-4	1.5 E-1	8.2 E-4
0.9	0.704994	0.705338	0.481116	0.706085	3.4 E-4	2.2 E-1	1.0 E-3
1.0	0.841471	0.841762	0.540302	0.842428	2.9 E-4	3.0 E-1	9.5 E-4



(a) Graph of exact and OHAM solutions



(b) Graph of exact and HPM solutions



(c) Graph of exact and HAM solutions

Fig. 3.1.

From the Fig. 3.1 (a) we investigate that the graphs of OHAM and exact solutions coincide while in Fig. 3.1 (b) the graphs of HPM and exact solutions are not coincident completely and the graphs of Fig. 3.1 (c) are again coinciding but generally the graphs of OHAM are in best agreement with their exact solutions. From the Table 3.1 given above we conclude that the exact and OHAM solutions are in best agreement in the domain $a = 0, b = 1$, solid curve elucidates exact solution, dotted curve provides OHAM solution. We conclude that the two curves coincide.

Problem 3.2:

For $y \in [0,1]$ we consider the following differential equation

$$F''(y) + 2F'(y) + F(y) = 0, F(0) = 1, F'(0) = 0, \tag{3.2 a}$$

The exact solution of the problem is

$$F(y) = e^{-y} + ye^{-y}, \tag{3.2 b}$$

Applying the technique, OHAM our zeroth order problem is

$$F_0''(y) + 2F_0'(y) + F_0(y) = 0, F_0(0) = 1, F_0'(0) = 0, \tag{3.2 c}$$

Its solution is

$$F_0(y) = \cos y, \tag{3.2 d}$$

First order problem is

$$F_1''(y,C_1) = -F_1(y) - 2F_1'(y) + (1+C_1)[F_0(y) + 2F_0'(y) + F_0''(y)], F_1(0) = 0, F_1'(0) = 0, \quad (3.2\text{ e})$$

Its solution is

$$F_1(y,C_1) = \frac{1}{2}\begin{pmatrix} 2y\cos y - \sin y + \cos 2y \sin y - \cos y \sin 2y + 2y\cos yC_1 - \sin yC_1 \\ + \cos 2y \sin yC_1 - \cos y \sin 2yC_1 \end{pmatrix}, \quad (3.2\text{ f})$$

Second order problem is

$$F_2''(y,C_1,C_2) = \begin{pmatrix} C_2F_0(y) - F_2(y) + 2C_2F_0'(y) - 2F_2'(y) + C_2F_0''(y) + (1+C_1)[F_1(y) + 2F_1'(y) + F_1''(y)] \\ F_2(0) = 0, F_2'(0) = 0 \end{pmatrix}, \quad (3.2\text{ g})$$

Its solution is

$$F_2(y,C_1,C_2) = \frac{1}{4}\begin{pmatrix} \cos y + 4y\cos y + 2y^2\cos y - \cos y\cos 2y - 2\sin y + 2\cos 2y\sin y + 2y\cos 2y\sin y - \\ 2\cos y\sin 2y - 2y\cos y\sin 2y - \sin y\sin 2y + 2\cos y\,C_1 + 8y\cos y\,C_1 + 4y^2\cos y\,C_1 - \\ 2\cos y\cos 2y\,C_1 - 4\sin y\,C_1 + 4\cos 2y\sin y\,C_1 + 4y\cos 2y\sin y\,C_1 - 4\cos y\sin 2y\,C_1 - \\ 4y\cos y\sin 2y\,C_1 - 2\sin y\sin 2y\,C_1 + \cos y\,C_1^2 + 4y\cos y\,C_1^2 + 2y^2\cos y\,C_1^2 - \\ \cos y\cos 2y\,C_1^2 - 2\sin y\,C_1^2 + 2\cos 2y\sin y\,C_1^2 + 2y\cos 2y\sin y\,C_1^2 - 2\cos y\sin 2y\,C_1^2 \\ -2y\cos y\sin 2y\,C_1^2 - \sin y\sin 2y\,C_1^2 + 4y\cos y\,C_2 - 2\sin y\,C_2 + 2\cos 2y\sin y\,C_2 - \\ 2\cos y\sin 2y\,C_2 \end{pmatrix}, \quad (3.2\text{ h})$$

Third order problem is

$$F_0''(y,C_1,C_2,C_3) = \begin{pmatrix} C_3F_0(y) + C_2F_1(y) - F_3(y) + 2C_3F_0'(y) + 2C_2F_1'(y) - 2F_3'(y) + C_3F_0''(y) + C_2F_1''(y) \\ (1+C_1)[F_2(y) + 2F_2'(y) + F_2''(y), F_3(0) = 0, F_3'(0) = 0 \end{pmatrix}, \quad (3.2\text{ i})$$

Its solution is

$$F_3(y,C_1,C_2,C_3) = \frac{1}{2}\begin{pmatrix} C_2F_0(y)[-\cos y - 2y\cos y + 2\cos^2 y - \cos y\cos 2y + \sin y - 2y\sin y + 2\cos^2 y\sin y - \\ \cos 2y\sin y + 2\sin^2 y - \sin y\sin 2y] + C_1C_2F_0(y)[-\cos y - 2y\cos y + 2\cos^2 y - \\ \cos y\cos 2y + \sin y - 2y\sin y + 2\cos^2 y\sin y - \cos 2y\sin y + 2\sin^2 y - \sin y\sin 2y] + \\ C_3F_0(y)[-2\cos y + 2\cos^2 y + 2\sin^2 y] + F_1(y)[-\cos y - 2y\cos y + 2\cos^2 y - \cos y\cos 2y \\ + \sin y - 2y\sin y + 2\cos^2 y\sin y - \cos 2y\sin y + 2\sin^2 y - \sin y\sin 2y] + C_1F_1(y)[-2\cos y \\ -4y\cos y + 4\cos^2 y + 2\cos y\cos 2y + 2\sin y - 4y\sin y + 4\cos^2 y\sin y - 2\cos 2y\sin y + \\ 4\sin^2 y - 2\sin y\sin 2y] + C_1^2F_1(y)[-\cos y - 2y\cos y + 2\cos^2 y - \cos y\cos 2y + \sin y - \\ 2y\sin y + 2\cos^2 y\sin y - \cos 2y\sin y + 2\sin^2 y - \sin y\sin 2y] + C_2F_1(y)[-2\cos y + \\ 2\cos^2 y + 2\sin^2 y] \end{pmatrix}, \quad (3.2\text{ j})$$

We use equations (3.2 d), (3.2 f), (3.2 h) and (3.2 j) we get third order approximate OHAM solution for $p=1$

$$\tilde{F}(y,C_1,C_2,C_3)=F_0(y)+F_1(y,C_1)+F_2(y,C_1,C_2)+F_3(y,C_1,C_2,C_3), \tag{3.2 k}$$

Using the OHAM technique of section mentioned above on domain $a=0, b=1$ we use the residual R that is

$$R=\tilde{F}''(y)+2\tilde{F}'(y)+\tilde{F}(y), \tag{3.2 l}$$

We have obtained the following values of C_1, C_2, C_3.where

$$C_1=-0.6157891011374226, C_2=-1.648024704278988, C_3=-0.48317914168990783, \tag{3.2 m}$$

From the above values of C_i's we get the following approximate OHAM solution

$$\tilde{F}(y)=1-0.424995\,y^2-0.161002\,y^3, \tag{3.2 n}$$

The HPM solution is

$$\tilde{F}(y)=1-0.5y^2+0.333333y^3+0.04167y^4+O(y^5), \tag{3.2 o}$$

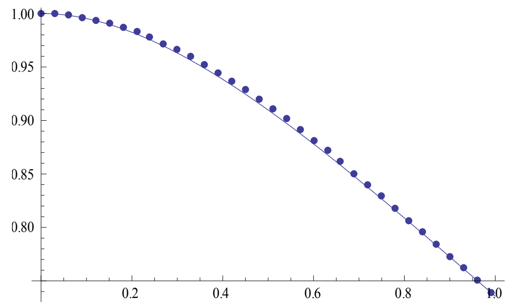
The HAM solution is

$$\tilde{F}(y)=1-0.464612y^2+0.209414y^3-0.00742224y^4-0.00671979y^5+O(y^6). \tag{3.2 p}$$

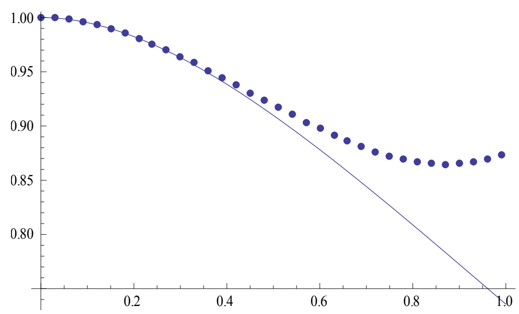
The following Table 3.2 displays values of the exact solution (3.2 b), OHAM solution (3.1 n) and Error of OHAM. We compare the two solutions there exists similarity approximations between them also the values of HPM, HAM along with their errors are considered and the comparison is established between the errors of OHAM, HPM and HAM the errors of the technique, OHAM are smaller than the other two. All the mentioned values are described in the following table 3.2, their graphs are drawn in Fig. 3.2 below.

Table 3.2.

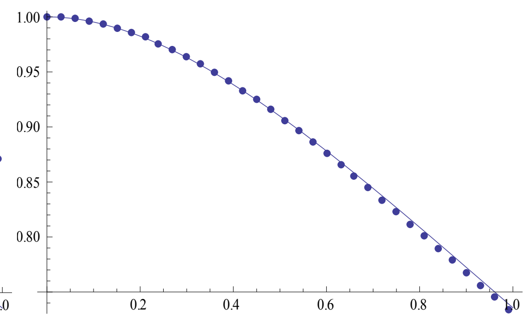
y	**Exact sol**	**OHAM sol**	**HPM sol**	**HAM sol**	**Er OHAM**	**Er HPM**	**Er HAM**
0.0	1.000000	1.000000	1.000000	1.000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	0.995321	0.99591	0.995338	0.995562	5.9 E-4	1.6 E-5	2.4 E-4
0.2	0.982477	0.984288	0.982733	0.983077	1.8 E-3	2.5 E-5	5.9 E-4
0.3	0.963064	0.966098	0.964338	0.963763	3.0 E-3	1.2 E-3	6.9 E-4
0.4	0.938448	0.942305	0.9424	0.938806	3.8 E-3	3.9 E-3	3.5 E-4
0.5	0.909796	0.923877	0.919271	0.90935	4.0 E-3	9.4 E-3	4.4 E-4
0.6	0.878099	0.881778	0.8974	0.876489	3.6 E-3	1.9 E-2	1.6 E-3
0.7	0.844195	0.846976	0.879337	0.841258	2.7 E-3	3.5 E-2	2.9 E-3
0.8	0.808792	0.810436	0.867733	0.804626	1.6 E-3	5.8 E-2	4.1 E-3
0.9	0,772482	0.773125	0.865338	0.76749	6.4 E-4	9.2 E-2	4.9 E-3
1.0	0.735759	0.736007	0.875	0.73066	2.4 E-4	1.3 E-1	5.0 E-3



(a) Graph of exact and OHAM solutions



(b) Graph of exact and HPM solution



(c) Graph of exact and HAM solutions

Fig. 3.2.

From the Fig. 3.2 (a) we conclude that the two graphs are coincident and also from the above Table: 3.2 We investigate that the OHAM solution and exact solution are in very good agreement on domain $a = 0, b = 1$ showing that the method is effective. Solid curve shows exact solution dotted curve shows OHAM solution, HPM solution and HAM solution. From figure 3.2(b) the HPM and exact solutions are not coincident again the HAM and exact solutions are coincident as shown in figure 3.2(c) , However generally the OHAM errors are smaller and there exists accuracy in the values of OHAM and the method is reliable and effective.

Problem 3.3:

For $y \in [0,1]$ we consider the following linear differential equation

$$F'''(y) + F(y) - (7 - y^2)\cos y - (y^2 - 6y - 1)\sin y = 0, F(0) = 0, F'(0) = -1, F'(1) = 2\sin(1), \quad (3.3\text{ a})$$

The exact solution of the problem is

$$F(y) = (y^2 - 1)\sin y, \quad (3.3\text{ b})$$

Applying the technique of OHAM that is described in the above section. The Zeroth order problem is

$$F_0'''(y) = 0, F_0(0) = 0, F_0'(0) = -1, F_1'(1) = 2\sin(1), \quad (3.3\text{ c})$$

Its solution is

$$F_0(y)=\frac{1}{2}\left(-2y+y^2(1+2\sin(1))\right), \tag{3.3 d}$$

First order problem is

$$F_1'''(y,C_1)=\begin{pmatrix} C_1[-7\cos y+y^2\cos y+\sin y+6y\sin y-y^2\sin y+F_0(y)]+(1+C_1)F_0'''(y) \\ F_1(0)=0,F_1'(0)=0,F_1'(1)=0 \end{pmatrix} \tag{3.3 e}$$

Its solution is

$$F_1(y,C_1)=\frac{1}{240}C_1\begin{pmatrix} -3120-240y+135y^2-10y^4+2y^5-480y^2\cos(1)+3120\cos y-240y^2\cos y+ \\ 950y^2\sin(1)+4y^5\sin(1)+240\sin y+1440y\sin y-240y^2\sin y \end{pmatrix}, \tag{3.3 f}$$

Second order problem is

$$F_2'''(y,C_1,C_2)=\begin{pmatrix} C_2[-7\cos y+y^2\cos y+\sin y+6y\sin y-y^2\sin y+F_0(y)+F_0'''(y)]+ \\ C_1F_1(y)+(1+C_1)F_1'''(y),F_2(0)=0,F_2'(0)=0,F_2'(1)=0 \end{pmatrix}, \tag{3.3 g}$$

Its solution is

$$F_2(y,C_1,C_2)=\frac{1}{40320}$$

$$\begin{pmatrix} C_1[-52416-40320y+22680y^2-1680y^4+336y^5-80640y^2\cos(1)+524160\cos y \\ -40320y^2\cos y+159600y^2\sin(1)+672y^5\sin(1)+40320\sin y+241920y\sin y- \\ 40320y^2\sin y]+C_1^2[-1048320+1209600y-468801y^2-87360y^3-3360y^4+714y^5- \\ 8y^7+y^8+446880y^2\cos(1)-1344y^5\cos(1)+1048320\cos y+483840y\cos y- \\ 80640y^2\cos y+475502y^2\sin(1)+3332y^5\sin(1)+2y^8\sin(1)-1693440\sin y+ \\ 483840y\sin y]+C_2[-524160-40320y+22680y^2-1680y^4+336y^5-80640y^2\cos(1)+ \\ 524160\cos y-40320y^2\cos y+159600y^2\sin(1)+672y^5\sin(1)+40320\sin y+ \\ 241920y\sin y-40320y^2\sin y] \end{pmatrix}, \tag{3.3 h}$$

Third order problem is

$$F_3'''(y,C_1,C_2,C_3)=$$

$$\begin{pmatrix} C_3[-7\cos y+y^2\cos y+\sin y+6y\sin y-y^2\sin y+F_0(y)+F_0'''(y)]+C_2[F_1(y)+F_1'''(y)]+ \\ C_1F_2(y)+(1+C_1)F_2'''(y),F_3(0)=0,F_3'(0)=0,F_3'(1)=0 \end{pmatrix}, \tag{3.3 i}$$

Its solution is obtained by using the method, OHAM. The optimal values of the auxiliary constants C_1, C_2 and C_3 are obtained by using Galerkin's method or least square method, using these values we get the series solution.

$$F_3(y,C_1,C_2,C_3) = \frac{1}{159667200}$$

$$\begin{pmatrix} C_1[-2075673600-159667200y+89812800y^2-6652800y^4+1330560y^5 \\ -319334400y^2\cos(1)+2075673600\cos y-159667200y^2\cos y+ \\ 632016000y^2\sin(1)+2661120y^5\sin(1)+159667200\sin y+958003200y\sin y \\ -159667200y^2\sin y]+C_1^2[-8302694400+9580032000y-3712903920y^2- \\ 691891200y^3-26611200y^4+5654880y^5-63360y^7+7920y^8+ \\ 3539289600y^2\cos(1)-10644480y^5\cos(1)+8302694400\cos y+ \\ 3832012800y\cos y-638668800y^2\cos y+3765975840y^2\sin(1)+ \\ 26389440y^5\sin(1)+15840y^8\sin(1)-13412044800\sin y+3832012800y\sin y] \\ +C_1^3[8302694400+14689382400y-6081277257y^2-1037836800y^3+ \\ 186278400y^4-28113426y^5-2882880y^6-95040y^7+12375y^8-44y^{10}+4y^{11} \\ +7443992160y^2\cos(1)+24171840y^5\cos(1)-15840y^8\cos(1)-8302694400\cos y \\ +5748019200y\cos y-319334400y^2\cos y-1867972634y^2\sin(1)+ \\ 44577852y^5\sin(1)+47190y^8\sin(1)+8y^{11}\sin(1)-20437401600\sin y \\ +319334400y^2\sin y]+C_2[-2075673600+89812800y^2-6652800y^4+1330560y^5 \\ -319334400y^2\cos(1)+2075673600\cos y-159667200y^2\cos y+ \\ 632016000y^2\sin(1)+2661120y^5\sin(1)+159667200\sin y+958003200y\sin y- \\ 159667200y^2\sin y]+C_1C_2[-8302694400+9580032000y-3712903920y^2- \\ 691891200y^3-26611200y^4+565880y^5-63360y^7+7920y^8+3539289600y^2\cos(1) \\ -10644480y^5\cos(1)+8302694400\cos y+3832012800y\cos y- \\ 638668800y^2\cos y+3765975840y^2\sin(1)+26389440y^5\sin(1)+15840y^8\sin(1)- \\ 13412044800\sin y+3822012800y\sin y]+C_3[-2075673600-159667200y+ \\ 89812800y^2-6652800y^4+1330560y^5-319334400y^2\cos(1)+2075673600\cos y+ \\ 159667200y^2\cos y+632016000y^2\sin(1)+2661120y^5\sin(1)+159667200\sin y+ \\ 958003200y\sin y-159667200y^2\sin y] \end{pmatrix}, \tag{3.3 j}$$

Now we use equations (3.3 d), (3.3 f), (3.3 h) and (3.3 j) to get third order approximate solution by OHAM for $p=1$ that is

$$\tilde{F}(y,C_1,C_2,C_3)=F_0(y)+F_1(y,C_1)+F_2(y,C_1,C_2)+F_3(y,C_1,C_2,C_3), \tag{3.3 k}$$

Using the proposed technique of section described above on the domain $a=0, b=1$ we use the residual

$$R=\tilde{F}'''(y)+\tilde{F}(y)-(7-y^2)\cos y-(y^2-6y-1)\sin y, \tag{3.3 l}$$

The following values of C_i's are found

$$C_1=1.062670102836802, C_2=-2.208103637790291, C_3=0.3336993779942651,$$

We use the above values of C_1, C_2, C_3, the approximate OHAM solution is

$$\tilde{F}(y) = -y + 0.0000488077y^2 + 1.16623y^3 - o.173884y^5 - 0.00092637y^6 + 0.00852856y^7 + 0.000200624y^8 - 0.000245363y^9 - 1.12908\times10^{-6}y^{10} + 4.06776\times10^{-6}y^{11} - 3.01577\times10^{-9}y^{12} - 4.10508\times10^{-8}y^{13} + 1.67752\times10^{-10}y^{14} + 2.89409\times10^{-10}y^{15} + O(y^{16}). \quad (3.3\text{ m})$$

The HPM solution is

$$\tilde{F}(y) = -y + 1.66667y^2 + 0.008532y^7 - 0.000220y^9 + O(y^{11}). \quad (3.3\text{ n})$$

The HAM solution is

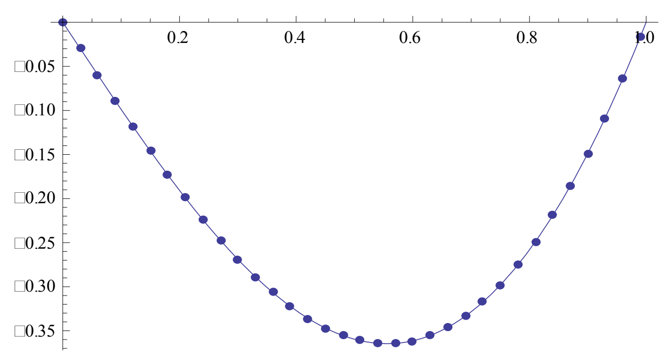
$$\tilde{F}(y) = -0.563288 + 32.3288y - 0.776059y^2 - 2.25957y^3 - 0.00180541y^4 + 0.026541y^5 + O(y^6), \quad (3.3\text{ o})$$

The following table displays the OHAM, HPM, HAM exact solutions and their errors.

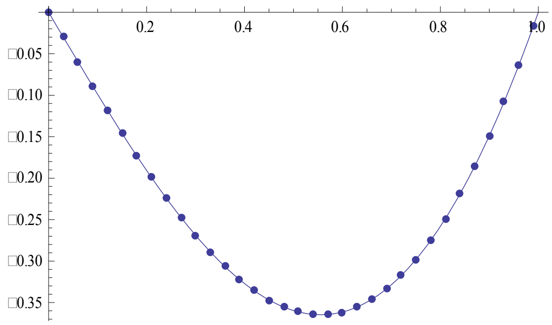
Table 3.3.

y	**Exact sol**	**OHAM sol**	**HPM sol**	**HAM sol**	**Er OHAM**	**Er HPM**	**Er HAM**
0.0	0.000000	0.000000	0.000000	0.000000	0.0 E-0	0.0 E-0	0.0 E-0
0.1	-0.0988351	-0.0886448	-0.0988249	-0.0988314	1.9 E-4	1.0 E-5	3.6 E-6
0.2	-0.190723	-0.190034	-0.190682	-0.19071	6.8 E-4	4.0 E-5	1.2 E-5
0.3	-0.268923	-0.267535	-0.268833	-0.268899	1.3 E-3	9.0 E-5	2.4 E-5
0.4	-0.327111	-0.32492	-0.326954	-0.327074	2.1 E-3	1.5 E-4	3.7 E-5
0.5	-0.359569	-0.357602	-0.35933	-0.35952	3.0 E-3	2.3 E-4	4.9 E-5
0.6	-0.361371	-0.357602	-0.361042	-0.36131	3.7 E-3	3.2 E-4	6.0 E-5
0.7	-0.328551	-0.324145	-0.328131	-0.32848	4.4 E-3	4.2 E-4	7.1 E-5
0.8	-0.258248	-0.253368	-0.257746	-0.258169	4.8 E-3	5.0 E-4	7.9 E-5
0.9	-0.148832	-0.143667	-0.148271	-0.148747	5.1 E-3	5.6 E-4	8.4 E-5
1.0	0.000000	0.0052575	0.00058439	0.0000869	5.2 E-3	5.8 E-4	8.6 E-5

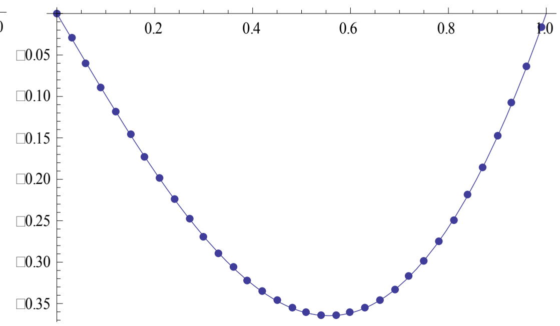
We conclude that the errors of technique, OHAM are smaller than HPM and HAM solutions. From the above Table 3.3 we conclude that OHAM and exact solutions are in best agreement and the values of the two columns are nearly equal.



(a) Graph of exact and OHAM solutions



(b) Graph of HPM and exact solutions



(c) Graph of HAM and exact solutions

Fig. 3.3.

In Fig. 3.3 (a), (b) and (c) we investigate that the graphs of OHAM, HPM and HAM are coincident with the graphs of their exact solutions, but generally we conclude that the technique, OHAM is more effective than the other two. From the above Fig. 3.3 we investigate that two graphs that is exact and OHAM solution graphs are coincident, this shows that the method OHAM is effective and reliable. Solid curve= exact solution, Dotted curve= OHAM solution, HPM solution and HAM solution.

4 Conclusion

The solutions of the problems 3.1, 3.2 and 3.3 are analyzed by using OHAM, HPM and HAM. The results are given in Tables 3.1, 3.2 and 3.3. Their graphs are displayed in Figs. 3.1, 3.2 and 3.3. Moreover, the solution of the problems are numerically and analytically by the above mentioned three techniques. These solutions are compared with their exact solutions and there exists the good agreement. We can apply the same techniques to all the initial and boundary value problems of all orders accordingly.

Acknowledgement

The authors are thankful to the anonymous reviewers and the Editor-in-Chief for the productive comments that led to positive enhancement in the paper.

Competing Interests

Authors have declared that no competing interests exist.

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