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Gaussian Generalized Tetranacci Numbers

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, we define Gaussian generalized Tetranacci numbers and as special cases, we investigate Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers with their properties. We present Binet's formulas, generating functions, and the summation formulas for Gaussian generalized Tetranacci numbers.Moreover, we give some identities connecting Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers. Furthermore, we present matrix formulation of Gaussian generalized Tetranacci numbers.

Keywords: Tetranacci numbers; Gaussian generalized Tetranacci numbers; Gaussian Tetranacci numbers; Gaussian Tetranacci-Lucas numbers.

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1 Introduction and Preliminaries

In this work, we define Gaussian generalized Tetranacci numbers and give properties of Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers as special cases. First, we present some background about generalized Tetranacci numbers and Gaussian numbers before defining Gaussian generalized Tetranacci numbers.

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There have been so many studies of the sequences of numbers in the literature which are defined recursively. Two of these type of sequences are the sequences of Tetranacci and Tetranacci-Lucas which are special case of generalized Tetranacci numbers. A generalized Tetranacci sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = V_{n-1} + V_{n-2} + V_{n-3} + V_{n-4}, \qquad (1.1)$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [1], [2], [3], [4], [5], [6].

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-1)} - V_{-(n-2)} - V_{-(n-3)} + V_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

The first few generalized Tetranacci numbers with positive subscript and negative subscript are given in the following Table 1:

Table 1. A few Tetranacci numbers

n	0	1	2	3	4	5	...
V_n	c_0	c_1	c_2	c_3	$c_0 + c_1 + c_2 + c_3$	$c_0 + 2c_1 + 2c_2 + 2c_3$	...
V_{-n}	c_0	$c_3 - c_2 - c_1 - c_0$	$2c_2 - c_3$	$2c_1 - c_2$	$2c_0 - c_1$	$2c_3 - 2c_2 - 2c_1 - 3c_0$	...

We consider two special cases of V_n : $V_n(0, 1, 1, 2) = M_n$ is the sequence of Tetranacci numbers (sequence A000078 in [7]) and $V_n(4, 1, 3, 7) = R_n$ is the sequence of Tetranacci-Lucas numbers (A073817 in [7]). In other words, Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \qquad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \qquad (1.2)$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \qquad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \qquad (1.3)$$

respectively.

Next, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts in the following Table 2:

Table 2. A few Tetranacci and Tetranacci-Lucas Numbers

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	...
M_{-n}	0	0	0	1	-1	0	0	2	-3	1	0	4	-8	5	...
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071	...
R_{-n}	4	-1	-1	-1	7	-6	-1	-1	15	-19	4	-1	31	-53	...

It is well known that for all integers n, usual Tetranaci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

(see for example [1] or [8])

or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1} \tag{1.4}$$

(see for example [9])

and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n$$

respectively, where α, β, γ and δ are the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\beta &= \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\gamma &= \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \\
\delta &= \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},
\end{aligned}$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

Note that we have the following identities:

$$\begin{aligned}
\alpha + \beta + \gamma + \delta &= 1, \\
\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta &= -1, \\
\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta &= 1, \\
\alpha\beta\gamma\delta &= -1.
\end{aligned}$$

We present an identity related with generalized Tetranacci numbers and Tetranacci numbers.

Theorem 1.1. For $n \geq 0$ and $m \geq 0$ the following identity holds:

$$V_{m+n} = M_{m-2}V_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})V_{n+2} + (M_{m-3} + M_{m-4})V_{n+1} + M_{m-3}V_n \tag{1.5}$$

Proof. We prove the identity by induction on m. If $m = 0$ then

$$V_n = M_{-2}V_{n+3} + (M_{-3} + M_{-4} + M_{-5})V_{n+2} + (M_{-3} + M_{-4})V_{n+1} + M_{-3}V_n$$

which is true because $M_{-2} = 0$, $M_{-3} = 1$, $M_{-4} = -1$, $M_{-5} = 0$. Assume that the equaliy holds for all $m \leq k$. For $m = k+1$, we have

$$\begin{aligned}
V_{(k+1)+n} &= V_{n+k} + V_{n+k-1} + V_{n+k-2} + V_{n+k-3} \\
&= (M_{k-2}V_{n+3} + (M_{k-3} + M_{k-4} + M_{k-5})V_{n+2} + (M_{k-3} + M_{k-4})V_{n+1} + M_{k-3}V_n) \\
&\quad +(M_{k-3}V_{n+3} + (M_{k-4} + M_{k-5} + M_{k-6})V_{n+2} + (M_{k-4} + M_{k-5})V_{n+1} + M_{k-4}V_n) \\
&\quad +(M_{k-4}V_{n+3} + (M_{k-5} + M_{k-6} + M_{k-7})V_{n+2} + (M_{k-5} + M_{k-6})V_{n+1} + M_{k-5}V_n) \\
&\quad +(M_{k-5}V_{n+3} + (M_{k-6} + M_{k-7} + M_{k-8})V_{n+2} + (M_{k-6} + M_{k-7})V_{n+1} + M_{k-6}V_n) \\
&= (M_{k-2} + M_{k-3} + M_{k-4} + M_{k-5})V_{n+3} \\
&\quad +((M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6}) + (M_{k-4} + M_{k-5} + M_{k-6} + M_{k-7}) \\
&\qquad\qquad + (M_{k-5} + M_{k-6} + M_{k-7} + M_{k-8}))V_{n+2} \\
&\quad +((M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6}) + (M_{k-4} + M_{k-5} + M_{k-6} + M_{k-7}))V_{n+1} \\
&\quad +(M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6})V_n \\
&= M_{k-1}V_{n+3} + (M_{k-2} + M_{k-3} + M_{k-4})V_{n+2} + (M_{k-2} + M_{k-3})V_{n+1} + M_{k-2}V_n \\
&= M_{(k+1)-2}V_{n+3} + (M_{(k+1)-3} + M_{(k+1)-4} + M_{(k+1)-5})V_{n+2} \\
&\quad +(M_{(k+1)-3} + M_{(k+1)-4})V_{n+1} + M_{(k+1)-3}V_n.
\end{aligned}$$

By induction on m, this proves (3.8).

The previous Theorem gives the following results as particular examples: For $n \geq 0$ and $m \geq 0$, we have (taking $V_n = M_n$)

$$M_{m+n} = M_{m-2}M_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})M_{n+2} + (M_{m-3} + M_{m-4})M_{n+1} + M_{m-3}M_n$$

and (taking $V_n = R_n$)

$$R_{m+n} = M_{m-2}R_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})R_{n+2} + (M_{m-3} + M_{m-4})R_{n+1} + M_{m-3}R_n.$$

Next we present the Binet's formula of the generalized Tetranacci sequence.

Lemma 1.2. The Binet's formula of the generalized Tetranacci sequence $\{V_n\}$ is given as

$$V_n = M_{n-3}V_0 + (M_{n-3} + M_{n-4})V_1 + (M_{n-3} + M_{n-4} + M_{n-5})V_2 + M_{n-2}V_3.$$

Proof. Take $n = 0$ and then replace n with m in Theorem 1.1

For another proof of the Lemma 1.2, see [4]. This Lemma is also a special case of a work on the nth k-generalized Fibonacci number (which is also called k-step Fibonacci number) in ([10], Theorem 2.2.).

Corollary 1.3. The Binet's formula of the generalized Tetranacci sequence $\{V_n\}$ is given as

$$V_n = A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\delta^{n-6}$$

where

$$\begin{aligned} A &= \frac{\alpha-1}{5\alpha-8}(V_3\alpha^3+(V_0+V_1+V_2)\alpha^2+(V_1+V_2)\alpha+V_2) \\ B &= \frac{\beta-1}{5\beta-8}(V_3\beta^3+(V_0+V_1+V_2)\beta^2+(V_1+V_2)\beta+V_2) \\ C &= \frac{\gamma-1}{5\gamma-8}(V_3\gamma^3+(V_0+V_1+V_2)\gamma^2+(V_1+V_2)\gamma+V_2) \\ D &= \frac{\delta-1}{5\delta-8}(V_3\delta^3+(V_0+V_1+V_2)\delta^2+(V_1+V_2)\delta+V_2) \end{aligned}$$

Proof. The proof follows from Lemma 1.2 and (1.4).

In fact, Corollary 1.3 is a special case of a result in ([10], Remark 2.3.).

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} a_n x^n$ of the sequence V_n.

Lemma 1.4. Suppose that $f_{V_n}(x) = \sum_{n=0}^{\infty} a_n x^n$ is the ordinary generating function of the generalized Tetranacci sequence $\{V_n\}_{n\geq 0}$.Then $f_{V_n}(x)$ is given by

$$f_{V_n}(x) = \frac{V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3}{1-x-x^2-x^3-x^4}. \tag{1.7}$$

Proof. Using (1.1) and some calculation, we obtain

$f_{V_n}(x)-xf_{V_n}(x)-x^2f_{V_n}(x)-x^3f_{V_n}(x)-x^4f_{V_n}(x) = V_0+(V_1-V_0)x+(V_2-V_1-V_0)x^2+(V_3-V_2-V_1-V_0)x^3$ which gives (1.7).

The previous Lemma gives the following results as particular examples: generating function of the Tetranacci sequence M_n is

$$f_{M_n}(x) = \sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-x-x^2-x^3-x^4}$$

and generating function of the Tetranacci-Lucas sequence R_n is

$$f_{R_n}(x) = \sum_{n=0}^{\infty} R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4}.$$

In literature, there have been so many studies of the sequences of Gaussian numbers. A Gaussian integer z is a complex number whose real and imaginary parts are both integers, i.e., $z = a + ib$, $a, b \in \mathbb{Z}$. These numbers is denoted by $\mathbb{Z}[i]$. The norm of a Gaussian integer $a + ib$, $a, b \in \mathbb{Z}$ is its Euclidean norm, that is, $N(a + ib) = \sqrt{a^2+b^2} = \sqrt{(a+ib)(a-ib)}$. For more information about this kind of integers, see the work of Fraleigh [11].

If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

In 1963, Horadam [12] introduced the concept of complex Fibonacci number called as the Gaussian Fibonacci number. Pethe [13] defined the complex Tribonacci numbers at Gaussian integers, see also [14]. There are other several studies dedicated to these sequences of Gaussian numbers such as the works in [15], [16], [17], [14], [18], [19], [20], [12], [21], [22], [23], [24], [25], [26], [27], among others.

2 Gaussian Generalized Tetranacci Numbers

Gaussian generalized Tetranacci numbers $\{GV_n\}_{n\geq 0} = \{GV_n(GV_0, GV_1, GV_2, GV_3)\}_{n\geq 0}$ are defined by

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4}, \tag{2.1}$$

with the initial conditions

$$GV_0 = c_0 + i(c_3 - c_2 - c_1 - c_0), GV_1 = c_1 + ic_0, GV_2 = c_2 + ic_1, GV_3 = c_3 + ic_2,$$

not all being zero. The sequences $\{GV_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$GV_{-n} = -GV_{-(n-1)} - GV_{-(n-2)} - GV_{-(n-3)} + GV_{-(n-4)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (2.1) hold for all integer n. Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1}. \tag{2.2}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}$$

The first few generalized Gaussian Tetranacci numbers with positive subscript and negative subscript are given in the following Table 3:

Table 3. A few generalized Gaussian Tetranacci numbers

n	GV_n	GV_{-n}
0	$c_0 + i(c_3 - c_2 - c_1 - c_0)$	$c_0 + i(c_3 - c_2 - c_1 - c_0)$
1	$c_1 + ic_0$	$(c_3 - c_2 - c_1 - c_0) + i(2c_2 - c_3)$
2	$c_2 + ic_1$	$2c_2 - c_3 + i(2c_1 - c_2)$
3	$c_3 + ic_2$	$2c_1 - c_2 + i(2c_0 - c_1)$
4	$c_0 + c_1 + c_2 + c_3 + ic_3$	$2c_0 - c_1 + i(2c_3 - 3c_0 - 2c_1 - 2c_2)$

We consider two special cases of $GV_n : GV_n(0, 1, 1+i, 2+i) = GM_n$ is the sequence of Gaussian Tetranacci numbers and $GV_n(4-i, 1+4i, 3+i, 7+3i) = GR_n$ is the sequence of Gaussian Tetranacci-Lucas numbers. We formally define them as follows:

Gaussian Tetranacci numbers are defined by

$$GM_n = GM_{n-1} + GM_{n-2} + GM_{n-3} + GM_{n-4}, \tag{2.3}$$

with the initial conditions

$$GM_0 = 0, GM_1 = 1, GM_2 = 1 + i, GM_3 = 2 + i$$

and Gaussian Tetranacci-Lucas numbers are defined by

$$GR_n = GR_{n-1} + GR_{n-2} + GR_{n-3} + GR_{n-4} \tag{2.4}$$

with the initial conditions

$$GR_0 = 4 - i, GR_1 = 1 + 4i, GR_2 = 3 + i, GR_3 = 7 + 3i.$$

Note that for $n \geq 0$

$$GM_n = M_n + iM_{n-1},\ GR_n = R_n + iR_{n-1}$$

and

$$GM_{-n} = M_{-n} + iM_{-n-1}, GR_{-n} = R_{-n} + iR_{-n-1}.$$

Next, we present the first few values of the Gaussian Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts in the following Table 4:

Table 4. A few Gaussian Tetranacci and Tetranacci-Lucas Numbers

n	0	1	2	3	4	5	6	7	8
GM_n	0	1	$1+i$	$2+i$	$4+2i$	$8+4i$	$15+8i$	$29+15i$	$56+29i$
GM_{-n}	0	0	i	$1-i$	-1	0	$2i$	$2-3i$	$-3+i$
GR_n	$4-i$	$1+4i$	$3+i$	$7+3i$	$15+7i$	$26+15i$	$51+26i$	$99+51i$	$191+99i$
GR_{-n}	$4-i$	$-1-i$	$-1-i$	$-1+7i$	$7-6i$	$-6-i$	$-1-i$	$-1+15i$	$15-19i$

The following Theorem presents the generating function of Gaussian generalized Tetranacci numbers.

THEOREM 2.1. The generating function of Gaussian generalized Tetranacci numbers is given as

$$f_{GV_n}(x) = \frac{GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3}{1 - x - x^2 - x^3 - x^4}. \quad (2.5)$$

Proof. Let

$$f_{GV_n}(x) = \sum_{n=0}^{\infty} GV_n x^n$$

be generating function of Gaussian generalized Tetranacci numbers. Then using the definition of generalized Gaussian Tetranacci numbers, and substracting $xf(x)$, $x^2f(x)$, $x^3f(x)$ and $x^4f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
& (1 - x - x^2 - x^3 - x^4) f_{GV_n}(x) \\
= & \sum_{n=0}^{\infty} GV_n x^n - x\sum_{n=0}^{\infty} GV_n x^n - x^2\sum_{n=0}^{\infty} GV_n x^n - x^3\sum_{n=0}^{\infty} GV_n x^n - x^4\sum_{n=0}^{\infty} GV_n x^n \\
= & \sum_{n=0}^{\infty} GV_n x^n - \sum_{n=0}^{\infty} GV_n x^{n+1} - \sum_{n=0}^{\infty} GV_n x^{n+2} - \sum_{n=0}^{\infty} GV_n x^{n+3} - \sum_{n=0}^{\infty} GV_n x^{n+4} \\
= & \sum_{n=0}^{\infty} GV_n x^n - \sum_{n=1}^{\infty} GV_{n-1} x^n - \sum_{n=2}^{\infty} GV_{n-2} x^n - \sum_{n=3}^{\infty} GV_{n-3} x^n - \sum_{n=4}^{\infty} GV_{n-4} x^n \\
= & (GV_0 + GV_1x + GV_2x^2 + GV_3x^3) - (GV_0x + GV_1x^2 + GV_2x^3) - (GV_0x^2 + GV_1x^3) - GV_0x^3 \\
& + \sum_{n=4}^{\infty} (GV_n - GV_{n-1} - GV_{n-2} - GV_{n-3} - GV_{n-4}) x^n \\
= & GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3
\end{aligned}$$

Rearranging above equation, we get

$$f_{GV_n}(x) = \frac{GV_0 + (GV_1 - GV_0)x + (GV_2 - GV_1 - GV_0)x^2 + (GV_3 - GV_2 - GV_1 - GV_0)x^3}{1 - x - x^2 - x^3 - x^4}.$$

The previous Theorem gives the following results as particular examples: the generating function of Gaussian Tetranacci numbers is

$$f_{GM_n}(x) = \frac{x + ix^2}{1 - x - x^2 - x^3 - x^4} \quad (2.6)$$

and the generating function of Gaussian Tetranacci-Lucas numbers is

$$f_{GR_n}(x) = \frac{-(1+i)x^3 - (2+2i)x^2 - (3-5i)x + 4 - i}{1 - x - x^2 - x^3 - x^4}. \tag{2.7}$$

The result (2.6) is already known, (see [25]).

We now present the Binet formula for the Gaussian generalized Tetranacci numbers.

THEOREM 2.2. The Binet formula for the Gaussian generalized Tetranacci numbers is

$$GV_n = \left(A\alpha^{n-6} + B\beta^{n-6} + C\gamma^{n-6} + D\gamma^{n-6}\right) + i\left(A\alpha^{n-7} + B\beta^{n-7} + C\gamma^{n-7} + D\gamma^{n-7}\right)$$

where A, B, C and D are as in Corollary (1.3).

Proof. The proof follows from Corollary (1.3) and $GV_n = V_n + iV_{n-1}$.

The previous Theorem gives the following results as particular examples: the Binet formula for the Gaussian Tetranacci numbers is

$$GM_n = \left(\begin{array}{c} \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ +\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{array}\right) + i\left(\begin{array}{c} \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ +\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \end{array}\right)$$

or

$$\begin{aligned} GM_n &= \left(\frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1}\right) \\ &\quad + \left(\frac{\alpha-1}{5\alpha-8}\alpha^{n-2} + \frac{\beta-1}{5\beta-8}\beta^{n-2} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-2} + \frac{\delta-1}{5\delta-8}\delta^{n-2}\right) \end{aligned}$$

and the Binet formula for the Gaussian Tetranacci-Lucas numbers is

$$GR_n = (\alpha^n + \beta^n + \gamma^n + \delta^n) + i\left(\alpha^{n-1} + \beta^{n-1} + \gamma^{n-1} + \delta^{n-1}\right).$$

The following Theorem present some formulas of Gaussian generalized Tetranacci numbers.

THEOREM 2.3. For $n \geq 1$ we have the following formulas:

(a) (Sum of the Gaussian generalized Tetranacci numbers)

$$\sum_{k=1}^{n} GV_k = \frac{1}{3}(GV_{n+2} + 2GV_n + GV_{n-1} - GV_0 + GV_1 - GV_3)$$

(b) $\sum_{k=1}^{n} GV_{2k+1} = \frac{1}{3}(2GV_{2n+2} + GV_{2n} - GV_{2n-1} - 2GV_0 - GV_1 - 3GV_2 + GV_3)$

(c) $\sum_{k=1}^{n} GV_{2k} = \frac{1}{3}(2GV_{2n+1} + GV_{2n-1} - GV_{2n-2} + GV_0 - GV_1 + 3GV_2 - 2GV_3)$.

Proof.

(a) Using the recurrence relation

$$GV_n = GV_{n-1} + GV_{n-2} + GV_{n-3} + GV_{n-4}$$

i.e.

$$GV_{n-4} = GV_n - GV_{n-1} - GV_{n-2} - GV_{n-3}$$

we obtain

$$\begin{aligned}
GV_0 &= GV_4 - GV_3 - GV_2 - GV_1 \\
GV_1 &= GV_5 - GV_4 - GV_3 - GV_2 \\
GV_2 &= GV_6 - GV_5 - GV_4 - GV_3 \\
GV_3 &= GV_7 - GV_6 - GV_5 - GV_4 \\
GV_4 &= GV_8 - GV_7 - GV_6 - GV_5 \\
&\vdots \\
GV_{n-4} &= GV_n - GV_{n-1} - GV_{n-2} - GV_{n-3} \\
GV_{n-3} &= GV_{n+1} - GV_n - GV_{n-1} - GV_{n-2} \\
GV_{n-2} &= GV_{n+2} - GV_{n+1} - GV_n - GV_{n-1} \\
GV_{n-1} &= GV_{n+3} - GV_{n+2} - GV_{n+1} - GV_n \\
GV_n &= GV_{n+4} - GV_{n+3} - GV_{n+2} - GV_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
\sum_{k=1}^{n} GV_k &= \frac{1}{3}(GV_{n+4} - GV_{n+2} - 2GV_{n+1} - GV_0 + GV_1 - GV_3) \\
&= \frac{1}{3}(GV_{n+2} + 2GV_n + GV_{n-1} - GV_0 + GV_1 - GV_3).
\end{aligned}$$

(b) When we use (2.1), we obtain the following equalities:

$$\begin{aligned}
GV_k &= GV_{k-1} + GV_{k-2} + GV_{k-3} + GV_{k-4} \\
GV_4 &= GV_3 + GV_2 + GV_1 + GV_0 \\
GV_6 &= GV_5 + GV_4 + GV_3 + GV_2 \\
GV_8 &= GV_7 + GV_6 + GV_5 + GV_4 \\
GV_{10} &= GV_9 + GV_8 + GV_7 + GV_6 \\
&\vdots \\
GV_{2n+2} &= GV_{2n+1} + GV_{2n} + GV_{2n-1} + GV_{2n-2}.
\end{aligned}$$

If we rearrange the above equalities, we obtain

$$\begin{aligned}
GV_3 &= GV_4 - GV_2 - GV_1 - GV_0 \\
GV_5 &= GV_6 - GV_4 - GV_3 - GV_2 \\
GV_7 &= GV_8 - GV_6 - GV_5 - GV_4 \\
GV_9 &= GV_{10} - GV_8 - GV_7 - GV_6 \\
&\vdots \\
GV_{2n-1} &= GV_{2n} - GV_{2n-2} - GV_{2n-3} - GV_{2n-4} \\
GV_{2n+1} &= GV_{2n+2} - GV_{2n} - GV_{2n-1} - GV_{2n-2}.
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
\sum_{k=1}^{n} GV_{2k+1} &= GV_{2n+2} - GV_2 - \sum_{k=1}^{2n-1} GV_k - GV_0 \\
&= GV_{2n+2} - GV_2 - \frac{1}{3}(GV_{(2n-1)+4} - GV_{(2n-1)+2} - 2GV_{(2n-1)+1} - GV_0 + GV_1 - GV_3) - GV_0 \\
&= GV_{2n+2} - GV_2 - \frac{1}{3}(GV_{2n+3} - GV_{2n+1} - 2GV_{2n} - GV_0 + GV_1 - GV_3) - GV_0 \\
&= -\frac{1}{3}(-3GV_{2n+2} + GV_{2n+3} - GV_{2n+1} - 2GV_{2n} + 2GV_0 + GV_1 + 3GV_2 - GV_3) \\
&= \frac{1}{3}(3GV_{2n+2} - GV_{2n+3} + GV_{2n+1} + 2GV_{2n} - 2GV_0 - GV_1 - 3GV_2 + GV_3)
\end{aligned}$$

and

$$\begin{aligned}
3GV_{2n+2} - GV_{2n+3} + GV_{2n+1} + 2GV_{2n} &= 2GV_{2n+2} + (GV_{2n+2} + GV_{2n+1} + GV_{2n} - GV_{2n+3}) + GV_{2n} \\
&= 2GV_{2n+2} + GV_{2n} - GV_{2n-1}
\end{aligned}$$

So

$$\sum_{k=1}^{n} GV_{2k+1} = \frac{1}{3}(2GV_{2n+2} + GV_{2n} - GV_{2n-1} - 2GV_0 - GV_1 - 3GV_2 + GV_3).$$

(c) Since

$$\sum_{k=1}^{n} GV_{2k+1} + \sum_{k=1}^{n} GV_{2k} = \sum_{k=1}^{2n+1} GV_k - GV_1$$

we have

$$\begin{aligned}
\sum_{k=1}^{n} GV_k &= \frac{1}{3}(GV_{n+4} - GV_{n+2} - 2GV_{n+1} - GV_0 + GV_1 - GV_3) \\
&= \frac{1}{3}(GV_{n+2} + 2GV_n + GV_{n-1} - GV_0 + GV_1 - GV_3), \\
\sum_{k=1}^{n} GV_{2k+1} &= \frac{1}{3}(2GV_{2n+2} + GV_{2n} - GV_{2n-1} - 2GV_0 - GV_1 - 3GV_2 + GV_3)
\end{aligned}$$

$$\begin{aligned}
\sum_{k=1}^{n} GV_{2k} &= \sum_{k=1}^{2n+1} GV_k - \sum_{k=1}^{n} GV_{2k+1} - GV_1 \\
&= \frac{1}{3}(GV_{(2n+1)+4} - GV_{(2n+1)+2} - 2GV_{(2n+1)+1} - GV_0 + GV_1 - GV_3) \\
&\quad -\frac{1}{3}(2GV_{2n+2} + GV_{2n} - GV_{2n-1} - 2GV_0 - GV_1 - 3GV_2 + GV_3) - GV_1 \\
&= \frac{1}{3}(GV_{2n+5} - GV_{2n+3} - 2GV_{2n+2} - GV_0 + GV_1 - GV_3) + \frac{1}{3}(-2GV_{2n+2} - GV_{2n} \\
&\quad +GV_{2n-1} + 2GV_0 + GV_1 + 3GV_2 - GV_3 - 3GV_1) \\
&= \frac{1}{3}(GV_{2n+5} - GV_{2n+3} - 2GV_{2n+2} - GV_0 + GV_1 - GV_3 - 2GV_{2n+2} - GV_{2n} + GV_{2n-1} \\
&\quad +2GV_0 + GV_1 + 3GV_2 - GV_3 - 3GV_1) \\
&= \frac{1}{3}(GV_{2n+5} - GV_{2n+3} - 4GV_{2n+2} - GV_{2n} + GV_{2n-1} + GV_0 - GV_1 + 3GV_2 - 2GV_3) \\
&= \frac{1}{3}(2GV_{2n+1} + GV_{2n-1} - GV_{2n-2} + GV_0 - GV_1 + 3GV_2 - 2GV_3)
\end{aligned}$$

This completes the proof.

As special cases of above Theorem, we have the following two Corollaries. First one present some formulas of Gaussian Tetranacci numbers.

COROLLARY 2.4. For $n \geq 1$ we have the following formulas:

(a) (Sum of the Gaussian Tetranacci numbers)

$$\sum_{k=1}^{n} GM_k = \frac{1}{3}(GM_{n+2} + 2GM_n + GM_{n-1} - (1+i))$$

(b) $\sum_{k=1}^{n} GM_{2k+1} = \frac{1}{3}(2GM_{2n+2} + GM_{2n} - GM_{2n-1} - 2 - 2i)$

(c) $\sum_{k=1}^{n} GM_{2k} = \frac{1}{3}(2GM_{2n+1} + GM_{2n-1} - GM_{2n-2} - 2 + i)$.

Second Corollary gives some formulas of Gaussian Tetranacci-Lucas numbers.

COROLLARY 2.5. For $n \geq 1$ we have the following formulas:

(a) (Sum of the Gaussian Tetranacci-Lucas numbers)

$$\sum_{k=1}^{n} GR_k = \frac{1}{3}(GR_{n+2} + 2GR_n + GR_{n-1} - 10 + 2i)$$

(b) $\sum_{k=1}^{n} GR_{2k+1} = \frac{1}{3}(2GR_{2n+2} + GR_{2n} - GR_{2n-1} - 11 - 2i)$

(c) $\sum_{k=1}^{n} GR_{2k} = \frac{1}{3}(2GR_{2n+1} + GR_{2n-1} - GR_{2n-2} - 2 - 8i)$.

In fact, using the method of the proof of Theorem 2.3, we can prove the following formulas of generalized Tetranacci numbers.

THEOREM 2.6. For $n \geq 1$ we have the following formulas:

(a) (Sum of the generalized Tetranacci numbers)

$$\sum_{k=1}^{n} V_k = \frac{1}{3}(V_{n+2} + 2V_n + V_{n-1} - V_0 + V_1 - V_3)$$

(b) $\sum_{k=1}^{n} V_{2k+1} = \frac{1}{3}(2V_{2n+2} + V_{2n} - V_{2n-1} - 2V_0 - V_1 - 3V_2 + V_3)$

(c) $\sum_{k=1}^{n} V_{2k} = \frac{1}{3}(2V_{2n+1} + V_{2n-1} - V_{2n-2} + V_0 - V_1 + 3V_2 - 2V_3)$.

As special cases of above Theorem, we have the following two Corollaries. First one present some formulas of Tetranacci numbers.

COROLLARY 2.7. For $n \geq 1$ we have the following formulas:

(a) (Sum of the Tetranacci numbers)

$$\sum_{k=1}^{n} M_k = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1)$$

(b) $\sum_{k=1}^{n} M_{2k+1} = \frac{1}{3}(2M_{2n+2} + M_{2n} - M_{2n-1} - 2)$

(c) $\sum_{k=1}^{n} M_{2k} = \frac{1}{3}(2M_{2n+1} + M_{2n-1} - M_{2n-2} - 2)$.

Second Corollary gives some formulas of Tetranacci-Lucas numbers.

COROLLARY 2.8. For $n \geq 1$ we have the following formulas:

(a) (Sum of the Tetranacci-Lucas numbers)

$$\sum_{k=1}^{n} R_k = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} - 10)$$

(b) $\sum_{k=1}^{n} R_{2k+1} = \frac{1}{3}(2R_{2n+2} + R_{2n} - R_{2n-1} - 11)$
(c) $\sum_{k=1}^{n} R_{2k} = \frac{1}{3}(2R_{2n+1} + R_{2n-1} - R_{2n-2} - 2)$.

Note that if the sum starts with the zero then the constant in the formula may only change, for example

$$\sum_{k=0}^{n} R_k = R_0 + \sum_{k=1}^{n} R_k = 4 + \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} - 10) = \frac{1}{3}(R_{n+2} + 2R_n + R_{n-1} + 2)$$

but

$$\sum_{k=0}^{n} M_k = M_0 + \sum_{k=1}^{n} M_k = \sum_{k=1}^{n} M_k = \frac{1}{3}(M_{n+2} + 2M_n + M_{n-1} - 1).$$

3 Some Identities Connecting Gaussian Tetranacci and Gaussian Tetranacci-Lucas Numbers

In this section, we obtain some identities of Gaussian Tetranacci numbers and Gaussian Tetranacci-Lucas numbers.

First, we can give a few basic relations between $\{GM_n\}$ and $\{GR_n\}$ as

$$GR_n = -GM_{n+3} + 6GM_{n+1} - GM_n \tag{3.1}$$

$$GR_n = -GM_{n+2} + 5GM_{n+1} - 2GM_n - GM_{n-1} \tag{3.2}$$

and also

$$GR_n = 4GM_{n+1} - 3GM_n - 2GM_{n-1} - GM_{n-2}. \tag{3.3}$$

Note that the last three identities hold for all integers n. For example, to show (3.1), writing

$$GR_n = aGM_{n+3} + bGM_{n+2} + cGM_{n+1} + dGM_n$$

and solving the system of equations

$$\begin{aligned} GR_0 &= aGM_3 + bGM_2 + cGM_1 + dGM_0 \\ GR_1 &= aGM_4 + bGM_3 + cGM_2 + dGM_1 \\ GR_2 &= aGM_5 + bGM_4 + cGM_3 + dGM_2 \\ GR_3 &= aGM_6 + bGM_5 + cGM_4 + dGM_3 \end{aligned}$$

we find that $a = -1, b = 0, c = 6, d = -1$. Or using the relations $GM_n = M_n + iM_{n-1}$, $GR_n = R_n + iR_{n-1}$ and identity $R_n = -M_{n+3} + 6M_{n+1} - M_n$ we obtain the identity (3.1). The others can be found similarly.

We will present some other identities between Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers with the help of generating functions.

The following lemma will help us to derive the generating functions of even and odd-indexed Gaussian Tetranacci and Gaussian Tetranacci-Lucas sequences.

LEMMA 3.1. [28] Suppose that $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ is the generating function of the sequence $\{a_n\}_{n\geq 0}$. Then the generating functions of the sequences $\{a_{2n}\}_{n\geq 0}$ and $\{a_{2n+1}\}_{n\geq 0}$ are given as

$$f_{a_{2n}}(x) = \sum_{n=0}^{\infty} a_{2n} x^n = \frac{f(\sqrt{x}) + f(-\sqrt{x})}{2}$$

and

$$f_{a_{2n+1}}(x) = \sum_{n=0}^{\infty} a_{2n+1}x^n = \frac{f(\sqrt{x}) - f(-\sqrt{x})}{2\sqrt{x}}$$

respectively.

The next Theorem presents the generating functions of even and odd-indexed generalized Tetranacci sequences.

THEOREM 3.2. The generating functions of the sequences V_{2n} and V_{2n+1} are given by

$$f_{V_{2n}}(x) = \frac{V_0 + (-3V_0 + V_2)x + (-2V_0 + V_1 - 2V_2 + V_3)x^2 + (-2V_2 + V_3)x^3}{x^4 + x^3 - 3x^2 - 3x + 1}$$

and

$$f_{V_{2n+1}}(x) = \frac{V_1 + (-3V_1 + V_3)x + (V_0 - V_1 + 2V_2 - V_3)x^2 + (V_0 + V_1 + V_2 - V_3)x^3}{x^4 + x^3 - 3x^2 - 3x + 1}$$

respectively.

Proof. Both statements are consequences of Lemma 3.1 applied to (1.7) and some lengthy work.

From the previous Theorem we get the following results as particular examples: the generating functions of the sequences M_{2n} and M_{2n+1} are given by

$$f_{M_{2n}}(x) = \frac{x^2 + x}{x^4 + x^3 - 3x^2 - 3x + 1}, \; f_{M_{2n+1}}(x) = \frac{-x^2 - x + 1}{x^4 + x^3 - 3x^2 - 3x + 1}$$

and the generating functions of the sequences R_{2n} and R_{2n+1} are given by

$$f_{R_{2n}}(x) = \frac{x^3 - 6x^2 - 9x + 4}{x^4 + x^3 - 3x^2 - 3x + 1}, \quad f_{R_{2n+1}}(x) = \frac{x^3 + 2x^2 + 4x + 1}{x^4 + x^3 - 3x^2 - 3x + 1}.$$

The next Theorem presents the generating functions of even and odd-indexed Gaussian generalized Tetranacci sequences.

THEOREM 3.3. The generating functions of the sequences GV_{2n} and GV_{2n+1} are given by

$$f_{GV_{2n}} = \frac{GV_0 + (-3GV_0 + GV_2)x + (-2GV_0 + GV_1 - 2GV_2 + GV_3)x^2 + (-2GV_2 + GV_3)x^3}{x^4 + x^3 - 3x^2 - 3x + 1}$$

and

$$f_{GV_{2n+1}} = \frac{GV_1 + (-3GV_1 + GV_3)x + (GV_0 - GV_1 + 2GV_2 - GV_3)x^2 + (GV_0 + GV_1 + GV_2 - GV_3)x^3}{x^4 + x^3 - 3x^2 - 3x + 1}$$

respectively.

Proof. Both statements are consequences of Lemma 3.1 applied to (2.5) and some lengthy algebraic calculations.

The previous theorem gives the following two corollaries as particular examples. Firstly, the next one presents the generating functions of even and odd-indexed Gaussian Tetranacci sequences.

COROLLARY 3.4. The generating functions of the sequences GM_{2n} and GM_{2n+1} are given by

$$f_{GM_{2n}} = \frac{(1+i)\,x + (1-i)\,x^2 - ix^3}{x^4 + x^3 - 3x^2 - 3x + 1} \tag{3.4}$$

and

$$f_{GM_{2n+1}} = \frac{1-(1-i)\,x-(1-i)\,x^2}{x^4+x^3-3x^2-3x+1} \tag{3.5}$$

respectively.

The following Corollary gives the generating functions of even and odd-indexed Gaussian Tetranacci-Lucas sequences.

COROLLARY 3.5. The generating functions of the sequences GR_{2n} and GR_{2n+1} are given by

$$f_{GR_{2n}}(x) = \frac{(4-i)-(9-4i)\,x-(6-7i)\,x^2+(1+i)\,x^3}{x^4+x^3-3x^2-3x+1} \tag{3.6}$$

and

$$f_{GR_{2n+1}}(x) = \frac{(1+4i)+(4-9i)\,x+(2-6i)\,x^2+(1+i)\,x^3}{x^4+x^3-3x^2-3x+1} \tag{3.7}$$

respectively.

The next Corollary present identities between Gaussian Tetranacci and Gaussian Tetranacci-Lucas sequences.

COROLLARY 3.6. We have the following identities:

$$\begin{aligned}
&(4-i)GM_{2n}-(9-4i)\,GM_{2n-2}-(6-7i)\,GM_{2n-4}+(1+i)\,GM_{2n-6}\\
=\;&(1+i)\,GR_{2n-2}+(1-i)\,GR_{2n-4}-iGR_{2n-6},\\
&(1+4i)GM_{2n}+(4-9i)\,GM_{2n-2}+(2-6i)\,GM_{2n-4}+(1+i)\,GM_{2n-6}\\
=\;&(1+i)\,GR_{2n-1}+(1-i)\,GR_{2n-3}-iGR_{2n-5}),\\
&(4-i)GM_{2n+1}-(9-4i)\,GM_{2n-1}-(6-7i)\,GM_{2n-3}+(1+i)\,GM_{2n-5}\\
=\;&GR_{2n}-(1-i)\,GR_{2n-2}-(1-i))GR_{2n-4}\\
&(1+4i)GM_{2n+1}+(4-9i)\,GM_{2n-1}+(2-6i)\,GM_{2n-3}+(1+i)\,GM_{2n-5}\\
=\;&GR_{2n+1}-(1-i)\,GR_{2n-1}-(1-i))GR_{2n-3}
\end{aligned}$$

Proof. From (3.4) and (3.6) we obtain

$$((4-i)-(9-4i)\,x-(6-7i)\,x^2+(1+i)\,x^3)f_{GM_{2n}} = ((1+i)\,x+(1-i)\,x^2-ix^3)f_{GR_{2n}}.$$

The LHS (left hand side) is equal to

$$\begin{aligned}
LHS &= ((4-i)-(9-4i)\,x-(6-7i)\,x^2+(1+i)\,x^3)\sum_{n=0}^{\infty}GM_{2n}x^n\\
&= (5-i)\,x^2+(5+3i)\,x+\sum_{n=3}^{\infty}((4-i)GM_{2n}-(9-4i)\,GM_{2n-2}\\
&\quad -(6-7i)\,GM_{2n-4}+(1+i)\,GM_{2n-6})x^n
\end{aligned}$$

whereas the RHS is

$$\begin{aligned}
RHS &= ((1+i)\,x+(1-i)\,x^2-ix^3)\sum_{n=0}^{\infty}GR_{2n}x^n\\
&= (5-i)\,x^2+(5+3i)\,x+\sum_{n=3}^{\infty}((1+i)\,GR_{2n-2}+(1-i)\,GR_{2n-4}-iGR_{2n-6})x^n.
\end{aligned}$$

Compare the coefficients and the proof of the first identity is done. The other identities can be proved similarly by using (3.4)-(3.7).

We present an identity related with Gaussian general Tetranacci numbers and Tetranacci numbers.

THEOREM 3.7. For $n \geq 0$ and $m \geq 0$ the following identity holds:

$$GV_{m+n} = M_{m-2}GV_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})GV_{n+2} + (M_{m-3} + M_{m-4})GV_{n+1} + M_{m-3}GV_n \quad (3.8)$$

Proof. We prove the identity by strong induction on m. If $m = 0$ then

$$GV_n = M_{-2}GV_{n+3} + (M_{-3} + M_{-4} + M_{-5})GV_{n+2} + (M_{-3} + M_{-4})GV_{n+1} + M_{-3}GV_n$$

which is true because $M_{-2} = 0, M_{-3} = 1, M_{-4} = -1, M_{-5} = 0,$. Assume that the equaliy holds for $m \leq k$. For $m = k + 1$, we have

$$\begin{aligned}
GV_{(k+1)+n} &= GV_{n+k} + GV_{n+k-1} + GV_{n+k-2} + GV_{n+k-3} \\
&= (M_{k-2}GV_{n+3} + (M_{k-3} + M_{k-4} + M_{k-5})GV_{n+2} + (M_{k-3} + M_{k-4})GV_{n+1} + M_{k-3}GV_n) \\
&\quad +(M_{k-3}GV_{n+3} + (M_{k-4} + M_{k-5} + M_{k-6})GV_{n+2} + (M_{k-4} + M_{k-5})GV_{n+1} + M_{k-4}GV_n) \\
&\quad +(M_{k-4}GV_{n+3} + (M_{k-5} + M_{k-6} + M_{k-7})GV_{n+2} + (M_{k-5} + M_{k-6})GV_{n+1} + M_{k-5}GV_n) \\
&\quad +(M_{k-5}GV_{n+3} + (M_{k-6} + M_{k-7} + M_{k-8})GV_{n+2} + (M_{k-6} + M_{k-7})GV_{n+1} + M_{k-6}GV_n) \\
&= (M_{k-2} + M_{k-3} + M_{k-4} + M_{k-5})GV_{n+3} \\
&\quad +((M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6}) + (M_{k-4} + M_{k-5} + M_{k-6} + M_{k-7}) \\
&\quad +(M_{k-5} + M_{k-6} + M_{k-7} + M_{k-8}))GV_{n+2} \\
&\quad +((M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6}) + (M_{k-4} + M_{k-5} + M_{k-6} + M_{k-7}))GV_{n+1} \\
&\quad +(M_{k-3} + M_{k-4} + M_{k-5} + M_{k-6})GV_n \\
&= M_{k-1}GV_{n+3} + (M_{k-2} + M_{k-3} + M_{k-4})GV_{n+2} + (M_{k-2} + M_{k-3})GV_{n+1} + M_{k-2}GV_n \\
&= M_{(k+1)-2}GV_{n+3} + (M_{(k+1)-3} + M_{(k+1)-4} + M_{(k+1)-5})GV_{n+2} \\
&\quad +(M_{(k+1)-3} + M_{(k+1)-4})GV_{n+1} + M_{(k+1)-3}GV_n
\end{aligned}$$

By strong induction on m, this proves (3.8).

The previous Theorem gives the following results as particular examples: For $n \geq 0$ and $m \geq 0$, we have (taking $GV_n = GM_n$)

$$GM_{m+n} = M_{m-2}GM_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})GM_{n+2} + (M_{m-3} + M_{m-4})GM_{n+1} + M_{m-3}GM_n$$

and (taking $GV_n = GR_n$)

$$GR_{m+n} = M_{m-2}GR_{n+3} + (M_{m-3} + M_{m-4} + M_{m-5})GR_{n+2} + (M_{m-3} + M_{m-4})GR_{n+1} + M_{m-3}GR_n.$$

4 Matrix Formulation of GV_n

Now, consider the sequence $\{U_n\}$ which is defined by the fourth-order recurrence relation

$$U_n = U_{n-1} + U_{n-2} + U_{n-3} + U_{n-4}, \qquad U_0 = U_1 = 0, U_2 = U_3 = 1.$$

Next, we present the first few values of numbers U_n with positive and negative subscripts in the following Table 5:

Table 5. A few values of the numbers U_n

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	...
U_n	0	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	...
U_{-n}	0	0	1	-1	0	0	2	-3	1	0	4	-8	5	-1	8	...

Note that some authors call $\{U_n\}$ as a Tetranacci sequence instead of $\{M_n\}$. The numbers U_n can be expressed using Binet's formula

$$U_n = \frac{\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.$$

The matrix method is very useful method in order to obtain some identities for special sequences. We define the square matrix A of order 4 as:

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det M = -1$. Induction proof may be used to establish

$$A^n = \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix} \quad (4.1)$$

$$= \begin{pmatrix} U_{n+2} & U_{n+1} + U_n + U_{n-1} & U_{n+1} + U_n & U_{n+1} \\ U_{n+1} & U_n + U_{n-1} + U_{n-2} & U_n + U_{n-1} & U_n \\ U_n & U_{n-1} + U_{n-2} + U_{n-3} & U_{n-1} + U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2} + U_{n-3} + U_{n-4} & U_{n-2} + U_{n-3} & U_{n-2} \end{pmatrix} \quad (4.2)$$

$$= \begin{pmatrix} U_{n+2} & U_{n+2} - U_{n-2} & U_{n+1} + U_n & U_{n+1} \\ U_{n+1} & U_{n+1} - U_{n-3} & U_n + U_{n-1} & U_n \\ U_n & U_n - U_{n-4} & U_{n-1} + U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-1} - U_{n-5} & U_{n-2} + U_{n-3} & U_{n-2} \end{pmatrix}. \quad (4.3)$$

Matrix formulation of M_n and R_n can be given as

$$\begin{pmatrix} M_{n+3} \\ M_{n+2} \\ M_{n+1} \\ M_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} M_3 \\ M_2 \\ M_1 \\ M_0 \end{pmatrix} \quad (4.4)$$

and

$$\begin{pmatrix} R_{n+3} \\ R_{n+2} \\ R_{n+1} \\ R_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} R_3 \\ R_2 \\ R_1 \\ R_0 \end{pmatrix}. \quad (4.5)$$

Induction proofs may be used to establish the matrix formulations M_n and R_n. Note that

$$GM_n = iU_n + U_{n+1}.$$

Consider the matrices N_M, E_M defined by as follows:

$$\begin{aligned} N_M &= \begin{pmatrix} 2+i & 1+i & 1 & 0 \\ 1+i & 1 & 0 & 0 \\ 1 & 0 & 0 & i \\ 0 & 0 & i & 1-i \end{pmatrix} \\ E_M &= \begin{pmatrix} GM_{n+3} & GM_{n+2} & GM_{n+1} & GM_n \\ GM_{n+2} & GM_{n+1} & GM_n & GM_{n-1} \\ GM_{n+1} & GM_n & GM_{n-1} & GM_{n-2} \\ GM_n & GM_{n-1} & GM_{n-2} & GM_{n-3} \end{pmatrix} \end{aligned}$$

Next Theorem presents the relations between A^n, N_M and E_M.

THEOREM 4.1. For $n \geq 3$, we have

$$A^n N_M = E_M.$$

Proof. Using the relation

$$GM_n = iU_n + U_{n+1},$$

and the calculations

$$\begin{aligned} a &= (2+i)\,U_n + (1+i)\,U_{n-1} + (2+i)\,U_{n+1} + (2+i)\,U_{n+2} \\ &= 2U_n + iU_n + U_{n-1} + iU_{n-1} + 2U_{n+1} + iU_{n+1} + 2U_{n+2} + iU_{n+2} \\ &= i(U_{n+2} + U_{n+1} + U_n + U_{n-1}) + (2U_{n+2} + 2U_{n+1} + 2U_n + U_{n-1}) \\ &= iU_{n+3} + (U_{n+2} + U_{n+1} + U_n + (U_{n+2} + U_{n+1} + U_n + U_{n-1})) \\ &= iU_{n+3} + (U_{n+2} + U_{n+1} + U_n + U_{n+3}) = iU_{n+3} + U_{n+4} = GM_{n+3} \end{aligned}$$

and

$$\begin{aligned} U_n + U_{n-1} + U_{n+1} + (1+i)\,U_{n+2} &= U_n + U_{n-1} + U_{n+1} + U_{n+2} + iU_{n+2} \\ &= iU_{n+2} + U_{n+3} = GM_{n+2}, \end{aligned}$$

we get

$$\begin{aligned} A^n N_M &= \begin{pmatrix} U_{n+2} & U_{n+1}+U_n+U_{n-1} & U_{n+1}+U_n & U_{n+1} \\ U_{n+1} & U_n+U_{n-1}+U_{n-2} & U_n+U_{n-1} & U_n \\ U_n & U_{n-1}+U_{n-2}+U_{n-3} & U_{n-1}+U_{n-2} & U_{n-1} \\ U_{n-1} & U_{n-2}+U_{n-3}+U_{n-4} & U_{n-2}+U_{n-3} & U_{n-2} \end{pmatrix} \begin{pmatrix} 2+i & 1+i & 1 & 0 \\ 1+i & 1 & 0 & 0 \\ 1 & 0 & 0 & i \\ 0 & 0 & i & 1-i \end{pmatrix} \\ &= \begin{pmatrix} GM_{n+3} & GM_{n+2} & GM_{n+1} & GM_n \\ GM_{n+2} & GM_{n+1} & GM_n & GM_{n-1} \\ GM_{n+1} & GM_n & GM_{n-1} & GM_{n-2} \\ GM_n & GM_{n-1} & GM_{n-2} & GM_{n-3} \end{pmatrix}. \end{aligned}$$

Above Theorem can be proved by mathematical induction as well.

Consider the matrices N_R, E_R defined by as follows:

$$\begin{aligned} N_R &= \begin{pmatrix} 7+3i & 3+i & 1+4i & 4-i \\ 3+i & 1+4i & 4-i & -1-i \\ 1+4i & 4-i & -1-i & -1-i \\ 4-i & -1-i & -1-i & -1+7i \end{pmatrix} \\ E_R &= \begin{pmatrix} GR_{n+3} & GR_{n+2} & GR_{n+1} & GR_n \\ GR_{n+2} & GR_{n+1} & GR_n & GR_{n-1} \\ GR_{n+1} & GR_n & GR_{n-1} & GR_{n-2} \\ GR_n & GR_{n-1} & GR_{n-2} & GR_{n-3} \end{pmatrix}. \end{aligned}$$

The following Theorem presents the relations between A^n, N_R and E_R.

THEOREM 4.2. We have

$$A^n N_R = E_R.$$

Proof. The proof requires some lengthy calculation, so we omit it.

The previous Theorem, also, can be proved by mathematical induction.

Similarly, matrix formulation of V_n can be given as

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}$$

Consider the matrices N_V, E_V defined by as follows:

$$\begin{aligned} N_V &= \begin{pmatrix} ic_2+c_3 & ic_1+c_2 & ic_0+c_1 & a_1 \\ ic_1+c_2 & ic_0+c_1 & (1-i)c_0-ic_1-ic_2+ic_3 & a_2 \\ ic_0+c_1 & (1-i)c_0-ic_1-ic_2+ic_3 & (1-i)c_3-c_1-(1-2i)c_2-c_0 & a_3 \\ (1-i)c_0-ic_1-ic_2+ic_3 & (1-i)c_3-c_1-(1-2i)c_2-c_0 & 2ic_1+(2-i)c_2-c_3 & a_4 \end{pmatrix}, \\ E_V &= \begin{pmatrix} GV_{n+3} & GV_{n+2} & GV_{n+1} & GV_n \\ GV_{n+2} & GV_{n+1} & GV_n & GV_{n-1} \\ GV_{n+1} & GV_n & GV_{n-1} & GV_{n-2} \\ GV_n & GV_{n-1} & GV_{n-2} & GV_{n-3} \end{pmatrix}. \end{aligned}$$

where

$$\begin{aligned} a_1 &= (1-i)c_0 - ic_1 - ic_2 + ic_3 \\ a_2 &= (1-i)c_3 - c_1 - (1-2i)c_2 - c_0 \\ a_3 &= 2ic_1 + (2-i)c_2 - c_3 \\ a_4 &= 2ic_0 + (2-i)c_1 - c_2. \end{aligned}$$

We now present our final Theorem.

THEOREM 4.3. We have

$$A^n N_V = E_V.$$

Proof. The proof requires some lengthy work, so we omit it.

5 Conclusions

Recently, there have been so many studies of the sequences of numbers in the literature that concern about subsequences of the Horadam numbers and generalized Fibonacci numbers such as Fibonacci, Lucas, Pell and Jacobsthal numbers; Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas numbers. If we use together sequences of integers defined recursively and Gaussian type integers, we obtain a new sequences of complex numbers such as Gaussian Fibonacci, Gaussian Lucas, Gaussian Pell, Gaussian Pell-Lucas and Gaussian Jacobsthal numbers; Gaussian Padovan and Gaussian Pell-Padovan numbers; Gaussian Tribonacci numbers.

This study proposes to introduce the concept of the Gaussian generalized Tetranacci numbers, and as special cases, Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers.

We can summarize the sections as follows:

- In the section (1), we present some background about generalized Tetranacci numbers and Gaussian numbers.
- In the section (2), we define Gaussian generalized Tetranacci numbers and as special cases, we investigate Gaussian Tetranacci and Gaussian Tetranacci-Lucas numbers with their properties such as the generating functions, Binet's formulas and sums formulas of these Gaussian numbers.
- In the section (3), we obtain some identities of Gaussian Tetranacci numbers and Gaussian Tetranacci-Lucas numbers.
- In the section (4), we give matrix formulation of Gaussian generalized Tetranacci numbers.

It is our intention to continue the study and explore some properties of this type of sequences, such as Gaussian Pentanacci numbers.

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Competing Interests

Author has declared that no competing interests exist.

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