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Fixed Point Theorem for $(\phi, \mathfrak{F})$ – Expansive Mappings in Cone b-Metric Spaces over Banach Algebra

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Authors' contributions

This work was carried out in collaboration between both authors. Author RJH designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Authors RJH and KM managed the analysis of the study. Author KM managed the literature searches. Both authors read and approved the final manuscript.

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Abstract

The class functions and are used in this paper to establish the notion of fixed point theorem on expansive mappings. The primary result is a generalization of the fixed point theorem for $(\phi, \mathfrak{F})$- expansive mappings on cone b-metric spaces over Banach algebra $\mathfrak{A}$. Investigated are the fixed point's criteria for existence and uniqueness. Additionally, provide an illustration.

Keywords: Cone b-Metric space; Banach algebra; expansive mapping.

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1 Introduction

In 1993, Czerwik [1,2] developed the idea of b-metrics, which expanded on conventional metric spaces. Wang first proposed the notion of expanded maps in 1984 [3]. The ideas of extended onto mappings on cone metric space are later established by Aage and Salunke. For expansive mapping, fixed point results have been achieved in [4] and [5]. The concept of cone metric space over Banach algebras was first developed by Liu and Xu [6], who replaced Banach spaces as the objective spaces of cone metric space with Banach algebras. Numerous researchers have published works on fixed point theorems in various types of metric spaces in [7,8,9]. In this study, using the class functions and, we provide a fixed point theorem for expansive mappings. A generalisation of the fixed point theorem for $(\phi, \mathfrak{F})$- expansive mappings on cone b-metric spaces over Banach algebra is presented in the main theorem.

Let $\mathfrak{A}$ be a real Banach algebra, i.e. $\mathfrak{A}$ is a real Banach space where a multiplication operation is specified, provided that it satisfies the criteria mentioned below: For all $\varsigma, \varrho, \mathfrak{z}, \in \mathfrak{A}, \mathfrak{d} \in$.

(i) $\varsigma(\varrho\mathfrak{z}) = (\varsigma\varrho)\mathfrak{z}$;
(ii) $\varsigma(\varrho + \mathfrak{z}) = \varsigma\varrho + \varsigma\mathfrak{z}\ and\ (\varsigma + \varrho)\mathfrak{z} = \varsigma\mathfrak{z} + \varrho\mathfrak{z}$;
(iii) $\mathfrak{d}(\varsigma\varrho) = (\mathfrak{d}\varsigma)\varrho = \varsigma(\mathfrak{d}\varrho)$;
(iv) $\| \varsigma\varrho \| \preccurlyeq \| \varsigma \| \| \varrho \|$.

We shall assume that the Banach algebra $\mathfrak{A}$ has a unit, i.e., a multiplicative identity $e \ni e\varsigma = \varsigma e = \varsigma, \forall \varsigma \in \mathfrak{A}$. If there is an inverse element $\varrho \in \mathfrak{A}$ that has the property that $\varsigma\varrho = \varrho\varsigma = e$, then that element $\varsigma \in \mathfrak{A}$ is said to be invertible. The inverse of ς is labelled by ς^{-1}.

Let $\mathfrak{A}$ be a real Banach algebra with a unit e and $\varsigma \in \mathfrak{A}$. If the spectral radius $\rho(\varsigma)$ of x is less than 1, that is

$$\rho(\varsigma) = \lim_{\mathfrak{n}\to\infty} \| \varsigma^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} = \inf_{n\geq 1} \| \varsigma^{\mathfrak{n}} \|^{\frac{1}{\mathfrak{n}}} \prec 1$$

then $e - \varsigma$ is invertible. Now,

$$(e - \varsigma)^{-1} = \sum_{i=0}^{\infty} \varsigma_i.$$

A subset $\mathfrak{P}$ of $\mathfrak{A}$ is called a cone of $\mathfrak{A}$ if

i. $\{\theta, e\} \subset \mathfrak{P}$,
ii. $\mathfrak{P}^2 = \mathfrak{P}\mathfrak{P} \subset \mathfrak{P}, \mathfrak{P} \cap (-\mathfrak{P}) = \{\theta\}$,
iii. $\alpha\mathfrak{P} + \beta\mathfrak{P} \subset \mathfrak{P}\ \forall\ \alpha, \beta \in \mathfrak{R}$,

For a given cone $\mathfrak{P} \subset \mathfrak{A}$, we define a partial ordering $\preccurlyeq$ with respect to $\mathfrak{P}$ by $\varsigma \preccurlyeq \varrho \Leftrightarrow \varrho - \varsigma \in \mathfrak{P}$;$\varsigma \prec \varrho$ will settle for $\varsigma \preccurlyeq \varrho$ and $\varsigma \neq \varrho$, while $\varsigma \ll \varrho$ settle for $\varrho - \varsigma \in int\mathfrak{P}$, where $int\mathfrak{P}$ labelled the interior of $\mathfrak{P}$. If $int\mathfrak{P} \neq \emptyset$, then $\mathfrak{P}$ is called a solid cone. Write $\| . \|$ as the norm of $\mathfrak{A}$. A cone $\mathfrak{P}$ is called normal should there be a number $\succ 0$, $\exists \forall \varsigma, \varrho \in \mathfrak{A}$, we have $\theta \preccurlyeq \varsigma \preccurlyeq \varrho \Longrightarrow \| \varsigma \| \preccurlyeq \mathfrak{M} \| \varrho \|$. The normal constant of $\mathfrak{P}$ is the least positive integer that satisfies the criteria above. Note that, for any normal cone $\mathfrak{P}$ we have $\mathfrak{M} \geq 1$. Here, we'll assume that $\mathfrak{A}$ is a real Banach algebra with a unit $e, \mathfrak{P}$ is a solid cone and $\preccurlyeq$ with respect to $\mathfrak{P}$.

2 Preliminaries

Lemma 2.1

(see [10]) If $\mathfrak{E}$ is a real Banach space with a cone $\mathfrak{P}$ and if $\mathfrak{d} \preccurlyeq \delta\mathfrak{d}$ with $\mathfrak{d} \in \mathfrak{P}$ and $0 \preccurlyeq \delta \prec 1$, then $\mathfrak{d} = \theta$.

Lemma 2.2

(see [11]) If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and if $\preccurlyeq \mathfrak{u} \ll \mathfrak{c}\ \forall\ \theta \ll \mathfrak{c}$, then $\mathfrak{u} = \theta$.

Lemma 2.3

(see [11]) Let $\mathfrak{P}$ be a cone in a Banach algebra $\mathfrak{A}$ and $\mathcal{K} \in \mathfrak{P}$ be a given vector. Let $\{\mathfrak{u}_\mathfrak{n}\}$ be a sequence in $\mathfrak{P}$. If $\forall\, \mathfrak{c}_1 \gg \theta, \exists\, \mathfrak{N}_1 \ni \mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_1\ \forall\, \mathfrak{n} \succ \mathfrak{N}_1$, then $\forall\, \mathfrak{c}_2 \gg \theta, \exists\, \mathfrak{N}_2 \ni \mathcal{K}\mathfrak{u}_\mathfrak{n} \ll \mathfrak{c}_2\ \forall\, \mathfrak{n} \succ \mathfrak{N}_2$.

Lemma 2.4

(see [11]) If $\mathfrak{E}$ is a real Banach space with a solid cone $\mathfrak{P}$ and $\{\varsigma_\mathfrak{n}\} \subset \mathfrak{P}$ is a sequence with $\| \varsigma_\mathfrak{n} \| \to 0$ as $\mathfrak{n} \to \infty$, then $\forall\, \theta \ll \mathfrak{c}, \exists\, \mathfrak{N} \in \mathbb{N} \ni \mathfrak{n} \succ \mathfrak{N}$ we have, $\varsigma_\mathfrak{n} \ll \mathfrak{c}\ i.e. \varsigma_\mathfrak{n}$ is a $\mathfrak{c}$-sequence.

Lemma 2.5

(see [12]) Let $\mathfrak{A}$ be a Banach algebra with a unit $e, \mathfrak{h}, \mathcal{K} \in \mathfrak{A}$. If $\mathfrak{h}$ commutes with $\mathcal{K}$, then $\rho(\mathfrak{h} + \mathcal{K}) \preccurlyeq \rho(\mathfrak{h}) + \rho(\mathcal{K}),\ \rho(\mathfrak{h}\mathcal{K}) \preccurlyeq \rho(\mathfrak{h})\rho(\mathcal{K})$.

Remark 2.6

(see [12]) If $\rho(\varsigma) \prec 1$, then $\| \varsigma_n \| \to 0$ as $\mathfrak{n} \to \infty$.

Definition 2.7

(see [13]) Let $\mathfrak{X}$ be a non-empty set, $\omega \succcurlyeq 1$ be a constant and $\mathfrak{A}$ be a over Banach algebra. A function $D_b: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ is said to be a cone b-metric provide that, for all $\varsigma, \varrho, \mathfrak{z} \in \mathfrak{X}$,

(d1) $D_b(\varsigma, \varrho) = 0 \Leftrightarrow \varsigma = \varrho$;
(d2) $D_b(\varsigma, \varrho) = D_b(\varrho, \varsigma)$;
(d3) $D_b(\varsigma, \mathfrak{z}) \preccurlyeq \omega[D_b(\varsigma, \varrho) + D_b(\varrho, \mathfrak{z})]$.

A pair $(\mathfrak{X}, D_b)$ is called a cone b-metric space over Banach algebra $\mathfrak{A}$.

Example 2.8

Let $\mathfrak{A} = C[\mathfrak{d}, \mathscr{b}]$ be the set of continuous functions on the interval $[\mathfrak{d}, \mathscr{b}]$ with the supermom norm. Define multiplication in the conventional way. Then $\mathfrak{A}$ is a Banach algebra with a unit 1. Set $\mathfrak{P} = \{\varsigma \in \mathfrak{A}: \varsigma(t) \succcurlyeq 0, \mathfrak{t} \in [\mathfrak{d}, \mathscr{b}]\}$ and $\mathfrak{X} = \mathfrak{R}$. Defined a mapping $D_b: \mathfrak{X} \times \mathfrak{X} \to \mathfrak{A}$ by $D_b(\varsigma, \varrho)(\mathfrak{t}) = |\varsigma - \varrho|^\mathfrak{p} e^\mathfrak{t}\ \forall\, \varsigma, \varrho \in \mathfrak{X}$, where $\mathfrak{p} \succ 1$ is a constant. This makes $(\mathfrak{X}, D_b)$ into a cone b-metric space over Banach algebra $\mathfrak{A}$ with the coefficient $\omega = 2^{\mathfrak{p}-1}$, but it is not a cone metric space over Banach algebra since the triangle inequality is not satisfied.

Definition 2.9

(see [13]) Let $(\mathfrak{X}, D_b)$ be a cone b-metric space over Banach algebra $\mathfrak{A}, \varsigma \in \mathfrak{X}$, let $\{\varsigma_\mathfrak{n}\}$ be a sequence in $\mathfrak{X}$. Then

1) $\{\varsigma_\mathfrak{n}\}$ converges to ς whenever for every $\mathfrak{c} \in \mathfrak{A}$ with $\theta \ll \mathfrak{c}$ there is natural number $\mathfrak{n}_0 \ni D_b(\varsigma_\mathfrak{n}, \varsigma) \ll \mathfrak{c}, \forall\, \mathfrak{n} \succcurlyeq \mathfrak{n}_0$. We indicate this by $\lim_{\mathfrak{n}\to\infty} \varsigma_\mathfrak{n} = \varsigma$.
2) $\{\varsigma_\mathfrak{n}\}$ is a Cauchy sequence whenever $\forall\, \mathfrak{c} \in \mathfrak{A}$ with $\theta \ll \mathfrak{c}$ there is natural number $\mathfrak{n}_0 \ni D_b(\varsigma_\mathfrak{n}, \varsigma_\mathfrak{m}) \ll \mathfrak{c}, \forall\, \mathfrak{n}, \mathfrak{m} \succcurlyeq \mathfrak{n}_0$.
3) $\{\mathfrak{X}, D_b\}$ is complete cone b-metric if every Cauchy sequence in $\mathfrak{X}$ is convergent.

Lemma 2.10

(see [12]) Let $\mathfrak{E}$ be a real Banach space with a solid cone $\mathfrak{P}$

1) If $\mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3 \in \mathfrak{E}$ and $\mathfrak{d}_1 \preccurlyeq \mathfrak{d}_2 \ll \mathfrak{d}_3$, then $\mathfrak{d}_1 \ll \mathfrak{d}_3$.
2) If $\mathfrak{d}_1 \in \mathfrak{P}$ and $\mathfrak{d}_1 \ll \mathfrak{d}_3$ for each $\mathfrak{d}_3 \gg \theta$, then $\mathfrak{d}_1 = \theta$.

Lemma 2.11

(see [14]) Let $\mathfrak{P}$ be a solid cone in a Banach algebra $\mathfrak{A}$. Suppose that $\mathfrak{h} \in \mathfrak{P}$ and $\{\varsigma_\mathfrak{n}\} \subset \mathfrak{P}$ is a $\mathfrak{c}$-sequence. Then $\{\mathfrak{h}\varsigma_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Proposition 2.12

(see [14]) Let $\mathfrak{P}$ be a solid cone in a Banach space $\mathfrak{A}$ and let $\{\varsigma_\mathfrak{n}\}, \{\varrho_\mathfrak{n}\} \subset \mathfrak{X}$ be sequence. If $\{\varsigma_\mathfrak{n}\}$ and $\{\varrho_\mathfrak{n}\}$ are $\mathfrak{c}$-sequences and $\alpha, \beta \succ 0$ then $\{\alpha\varsigma_\mathfrak{n} + \beta\varrho_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Proposition 2.13

(see [14]) Let $\mathfrak{P}$ be a solid cone in a Banach algebra $\mathfrak{A}$ and let $\{\varsigma_\mathfrak{n}\} \subset \mathfrak{P}$ is a sequence. Such that

1) $\{\varsigma_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.
2) For each $\mathfrak{c} \gg \theta \ \exists\ \mathfrak{n}_0 \in \mathbb{N}\ \ni \varsigma_\mathfrak{n} \prec \mathfrak{c}$ for $\mathfrak{n} \succcurlyeq \mathfrak{n}_0$.
3) For each $\mathfrak{c} \gg \theta \ \exists\ \mathfrak{n}_1 \in \mathbb{N}\ \ni \varsigma_\mathfrak{n} \prec \mathfrak{c}$ for $\mathfrak{n} \succcurlyeq \mathfrak{n}_1$.

Lemma 2.14

(see [12]) Let $\mathfrak{A}$ be a Banach algebra with a unite $e, \mathfrak{h} \in \mathfrak{A}$, then $\lim\limits_{\mathfrak{n}\to\infty} \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}}$ exists and the spectral radius $\rho(\mathfrak{h})$ satisfies

$$\rho(\mathfrak{h}) = \lim_{\mathfrak{n}\to\infty} \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}} = \inf \| \mathfrak{h}^\mathfrak{n} \|^{\frac{1}{\mathfrak{n}}}.$$

If $\rho(\mathfrak{h}) \prec |\delta|$, then $(\delta e - \mathfrak{h})$ is invertible in $\mathfrak{A}$, now,

$$(\delta e - \mathfrak{h})^{-1} = \sum_{i=0}^{\infty} \frac{\mathfrak{h}^i}{\delta^{i+1}} \ \ and\ \ \rho((\delta e - \mathfrak{h})^{-1}) \preccurlyeq \frac{1}{|\delta| - \rho(\mathfrak{h})}$$

where δ is a constant.

Lemma 2.15

(see [12]) Let $\mathfrak{A}$ be a Banach algebra with a unit e and $\mathfrak{P}$ be a solid cone in $\mathfrak{A}$. Let $\mathfrak{h} \in \mathfrak{A}$ and $\varsigma_\mathfrak{n} = \mathfrak{h}^\mathfrak{n}$. If $\rho(\mathfrak{h}) \prec 1$, then $\{\varsigma_\mathfrak{n}\}$ is a $\mathfrak{c}$-sequence.

Definition 2.16

(see [13]) Let $(\mathfrak{X}, D_b)$ be a complete cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be a cone in $\mathfrak{A}$ with the coefficient $\omega \succcurlyeq 1$. Then $\xi: \mathfrak{X} \to \mathfrak{X}$ is said to be generalized expansive mapping, if

$$D_b(\xi\varsigma, \xi\varrho) \succcurlyeq \mathfrak{h} D_b(\varsigma, \varrho)$$

For all $\varsigma, \varrho \in \mathfrak{X}$, where $\mathfrak{h}, \mathfrak{h}^{-1} \in \mathfrak{P}$ are generalized constants with $\rho(\mathfrak{h}^{-1}) \prec 1$.

Definition 2.17

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P} = \mathfrak{R}_0^+$ be a cone in $\mathfrak{A}$. A mapping $\mathfrak{F}: \mathfrak{P} \to \mathfrak{P}$ such that

1) $\mathfrak{F}$ is non-decreasing and continuous;
2) $\lim_{\mathfrak{n}\to\infty} \mathfrak{F}^\mathfrak{n}(\mathfrak{t}) = \theta$ for all $(\mathfrak{t} \in \mathfrak{P}), \mathfrak{t} \succcurlyeq 0$, where $\mathfrak{F}^\mathfrak{n}$ stands for the $\mathfrak{n}^{th}$ iterate of $\mathfrak{F}$;
3) $\mathfrak{F}(\mathfrak{t}) \prec \mathfrak{t}\ \forall\ \mathfrak{t} \succ 0$;
4) $\mathfrak{F}(\theta) = \theta$.

Definition 2.18

Let $\mathfrak{A}$ be a Banach algebra and $\mathfrak{P} = \Re_0^+$ be a cone in $\mathfrak{A}$. A mapping $\phi: \mathfrak{P} \to \mathfrak{P}$ such that:

1) ϕ is monotone non-decreasing and continuous $i.e., \theta \preccurlyeq \mathrm{t}_1 \preccurlyeq \mathrm{t}_2 \Longrightarrow \phi(\mathrm{t}_1) \preccurlyeq \phi(\mathrm{t}_2)$;
2) $\{\phi^{\mathfrak{n}}(\mathrm{t})\}$ $(\mathrm{t} \succ 0)$ is a $\mathfrak{c}$-sequence in $\mathfrak{P}$;
3) If $\{\mathfrak{u}_n\}$ is a $\mathfrak{c}$-sequence in $\mathfrak{P}$, then $\{\phi(\mathfrak{u}_{\mathfrak{n}})\}$ is also a $\mathfrak{c}$-sequence in $\mathfrak{P}$;
4) $\phi(\mathrm{t}) = \mathcal{K}\mathrm{t}$, for some $(\mathcal{K} \in \mathfrak{P})$, $\mathcal{K} \succ 0$.

3 Main Results

We prove a unique fixed point for generalized $(\phi, \mathfrak{F})$-expansive mappings via the class functions Φ and Ψ.

Definition 3.1

Let $(\mathfrak{X}, D_b)$ be a cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P} = \Re_0^+$ be a cone in $\mathfrak{A}$ with constant $\omega \succcurlyeq 1$. Let $\xi: \mathfrak{X} \to \mathfrak{X}$ be define a generalised $(\phi, \mathfrak{F})$-expansive mappings if there exist two functions $\mathfrak{F}: \mathfrak{P} \to \mathfrak{P}$ and $\phi: \mathfrak{P} \to \mathfrak{P}$ where $\mathfrak{F} \in \Psi$ and $\phi \in \Phi$ such that

$$\phi(D_b(\xi\varsigma, \xi\varrho)) \succcurlyeq \mathfrak{F}\Big(\phi\big(\mathfrak{h}\, D_b(\varsigma, \varrho)\big)\Big),$$

for all $\varsigma, \varrho \in \mathfrak{X}$, where $\mathfrak{h}, \mathfrak{h}^{-1} \in \mathfrak{P}$ are generalized constants with $\rho(\mathfrak{h}^{-1}) \prec 1$.

Theorem 3.2

Let $(\mathfrak{X}, D_b)$ be a complete cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be an underlying solid cone in $\mathfrak{A}$ with the coefficient $\omega \succcurlyeq 1$. Let mapping $\xi: \mathfrak{X} \to \mathfrak{X}$ be a surjective self-map of $\mathfrak{X}$ satisfies the expansive condition

$$\phi(D_b(\xi\varsigma, \xi\varrho)) \succcurlyeq \mathfrak{F}(\phi(\mathcal{M}(\varsigma, \varrho)) \tag{3.1}$$

where,

$$\mathcal{M}(\varsigma, \varrho) = \alpha_1 \frac{D_b(\varsigma, \xi\varsigma)}{1 + D_b(\varsigma, \xi\varsigma)} + \alpha_2 \frac{D_b(\varrho, \xi\varrho)}{1 + D_b(\varrho, \xi\varrho)} + \alpha_3 \frac{D_b(\varsigma, \xi\varrho)}{1 + D_b(\varsigma, \xi\varrho)} + \alpha_4 \frac{D_b(\varrho, \xi\varsigma)}{1 + D_b(\varrho, \xi\varsigma)} + \alpha_5 D_b(\varsigma, \varrho)$$

for all $\varsigma, \varrho \in \mathfrak{X}$, where $\alpha_i \in \mathfrak{P}$, $(i = 1, 2, 3, 4, 5)$ such that $(\alpha_2 + \omega\alpha_4 + \alpha_5)^{-1}$, $(\alpha_1\omega + \alpha_3\omega + \alpha_5)^{-1} \in \mathfrak{P}$ and spectral radius $\rho\big((e - \alpha_1 - \omega\alpha_4)(\alpha_2 + \omega\alpha_4 + \alpha_5)^{-1}\big) \prec 1$. Then ξ has a unique fixed point.

Proof. Let $\varsigma_0 \in \mathfrak{X}$. Since ξ is surjective, there exists $\varsigma_1 \in \mathfrak{X}$ such that $\xi\varsigma_1 = \varsigma_0$.

$$\phi\big(D_b(\varsigma_0, \varsigma_1)\big) = \phi\big(D_b(\xi\varsigma_1, \xi\varsigma_2)\big)$$

$$\phi\big(D_b(\xi\varsigma_1, \xi\varsigma_2)\big) \succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1 \dfrac{D_b(\varsigma_1, \xi\varsigma_1)}{1 + D_b(\varsigma_1, \xi\varsigma_1)} + \alpha_2 \dfrac{D_b(\varsigma_2, \xi\varsigma_2)}{1 + D_b(\varsigma_2, \xi\varsigma_2)} + \alpha_3 \dfrac{D_b(\varsigma_1, \xi\varsigma_2)}{1 + D_b(\varsigma_1, \xi\varsigma_2)} \\ +\alpha_4 \dfrac{D_b(\varsigma_2, \xi\varsigma_1)}{1 + D_b(\varsigma_2, \xi\varsigma_1)} + \alpha_5 D_b(\varsigma_1, \varsigma_2)\end{array}\right)\right)$$

$$= \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1 \dfrac{D_b(\varsigma_1, \varsigma_0)}{1 + D_b(\varsigma_1, \varsigma_0)} + \alpha_2 \dfrac{D_b(\varsigma_2, \varsigma_1)}{1 + D_b(\varsigma_2, \varsigma_1)} + \alpha_3 \dfrac{D_b(\varsigma_1, \varsigma_1)}{1 + D_b(\varsigma_1, \varsigma_1)} \\ +\alpha_4 \dfrac{D_b(\varsigma_2, \varsigma_0)}{1 + D_b(\varsigma_2, \varsigma_0)} + \alpha_5 D_b(\varsigma_1, \varsigma_2)\end{array}\right)\right)$$

$$\phi\big(D_b(x_0, x_1)\big) \succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1 D_b(\varsigma_1, \varsigma_0) + \alpha_2 D_b(\varsigma_2, \varsigma_1) + \alpha_3 D_b(\varsigma_1, \varsigma_1) \\ +\alpha_4 D_b(\varsigma_2, \varsigma_0) + \alpha_5 D_b(\varsigma_1, \varsigma_2)\end{pmatrix}\right)$$

$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_1,\varsigma_2)+\alpha_1 D_b(\varsigma_1,\varsigma_0)\\ +\alpha_4\omega[D_b(\varsigma_0,\varsigma_1)+D_b(\varsigma_1,\varsigma_2)]\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_1,\varsigma_2)+\alpha_1 D_b(\varsigma_1,\varsigma_0)\\ +\alpha_4\omega D_b(\varsigma_0,\varsigma_1)+\alpha_4\omega D_b(\varsigma_1,\varsigma_2)\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)D_b(\varsigma_1,\varsigma_2)\\ +(\alpha_1+\omega\alpha_4)D_b(\varsigma_1,\varsigma_0)\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_1,\varsigma_2)\\ +(\alpha_1+\omega\alpha_4)\phi D_b(\varsigma_1,\varsigma_0)\end{pmatrix}$$

Which implies that

$$(e-\alpha_1-\omega\alpha_4)\,\phi\big(D_b(\varsigma_0,\varsigma_1)\big)\succcurlyeq\mathfrak{F}\big((\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_1,\varsigma_2)\big).$$

Again we can choose $\varsigma_2\in\mathfrak{X}$ such that $\xi\varsigma_2=\varsigma_1$. Now,

$$\phi\big(D_b(\varsigma_1,\varsigma_2)\big)=\phi\big(D_b(\xi\varsigma_2,\xi\varsigma_3)\big)$$
$$\phi\big(D_b(\xi\varsigma_2,\xi\varsigma_3)\big)\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma_2,\xi\varsigma_2)}{1+D_b(\varsigma_2,\xi\varsigma_2)}+\alpha_2\frac{D_b(\varsigma_3,\xi\varsigma_3)}{1+D_b(\varsigma_3,\xi\varsigma_3)}+\alpha_3\frac{D_b(\varsigma_2,\xi\varsigma_3)}{1+D_b(\varsigma_2,\xi\varsigma_3)}\\ +\alpha_4\frac{D_b(\varsigma_3,\xi\varsigma_2)}{1+D_b(\varsigma_3,\xi\varsigma_2)}+\alpha_5 D_b(\varsigma_2,\varsigma_3)\end{pmatrix}\right)$$
$$=\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma_2,\varsigma_1)}{1+D_b(\varsigma_2,\varsigma_1)}+\alpha_2\frac{D_b(\varsigma_3,\varsigma_2)}{1+D_b(\varsigma_3,\varsigma_2)}+\alpha_3\frac{D_b(\varsigma_2,\varsigma_2)}{1+D_b(\varsigma_2,\varsigma_2)}\\ +\alpha_4\frac{D_b(\varsigma_3,\varsigma_1)}{1+D_b(\varsigma_3,\varsigma_1)}+\alpha_5 D_b(\varsigma_2,\varsigma_3)\end{pmatrix}\right)$$
$$\phi\big(D_b(\varsigma_1,\varsigma_2)\big)\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1 D_b(\varsigma_2,\varsigma_1)+\alpha_2 D_b(\varsigma_3,\varsigma_2)+\alpha_3 D_b(\varsigma_2,\varsigma_2)\\ +\alpha_4 D_b(\varsigma_3,\varsigma_1)+\alpha_5 D_b(\varsigma_2,\varsigma_3)\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_3,\varsigma_2)+\alpha_1 D_b(\varsigma_2,\varsigma_1)\\ +\alpha_4\omega[D_b(\varsigma_1,\varsigma_2)+D_b(\varsigma_2,\varsigma_3)]\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_3,\varsigma_2)+\alpha_1 D_b(\varsigma_2,\varsigma_1)\\ +\alpha_4\omega D_b(\varsigma_1,\varsigma_2)+\alpha_4\omega D_b(\varsigma_2,\varsigma_3)\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)D_b(\varsigma_3,\varsigma_2)\\ +(\alpha_1+\omega\alpha_4)D_b(\varsigma_2,\varsigma_1)\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_3,\varsigma_2)\\ +(\alpha_1+\omega\alpha_4)\phi D_b(\varsigma_2,\varsigma_1)\end{pmatrix}$$

Which implies that

$$(e-\alpha_1-\omega\alpha_4)\,\phi\big(D_b(\varsigma_1,\varsigma_2)\big)\succcurlyeq\mathfrak{F}\big((\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_2,\varsigma_3)\big).$$

Continuing this process, we obtain by induction a sequence $\{\varsigma_n\}$ in $(\mathfrak{X},D_b)$ by $\varsigma_\mathfrak{n}=\xi\varsigma_{\mathfrak{n}+1}$, for $\mathfrak{n}=0,1,2,\ldots$

$$\phi\big(D_b(\varsigma_\mathfrak{n},\varsigma_{\mathfrak{n}+1})\big)=\phi\big(D_b(\xi\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+2})\big)$$
$$\phi\big(D_b(\xi\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+2})\big)\succcurlyeq$$
$$\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+1})}+\alpha_2\frac{D_b(\varsigma_{\mathfrak{n}+2},\xi\varsigma_{\mathfrak{n}+2})}{1+D_b(\varsigma_{\mathfrak{n}+2},\xi\varsigma_{\mathfrak{n}+2})}+\alpha_3\frac{D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+2})}{1+D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+2})}\\ +\alpha_4\frac{D_b(\varsigma_{\mathfrak{n}+2},\xi\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma_{\mathfrak{n}+2},\xi\varsigma_{\mathfrak{n}+1})}+\alpha_5 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\end{pmatrix}\right)$$
$$=\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma_{\mathfrak{n}+1},\varsigma_\mathfrak{n})}{1+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_\mathfrak{n})}+\alpha_2\frac{D_b(\varsigma_{\mathfrak{n}+2},\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma_{\mathfrak{n}+2},\varsigma_{\mathfrak{n}+1})}+\alpha_3\frac{D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+1})}\\ +\alpha_4\frac{D_b(\varsigma_{\mathfrak{n}+2},\varsigma_\mathfrak{n})}{1+D_b(\varsigma_{\mathfrak{n}+2},\varsigma_\mathfrak{n})}+\alpha_5 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\end{pmatrix}\right)$$
$$\phi\big(D_b(\varsigma_\mathfrak{n},\varsigma_{\mathfrak{n}+1})\big)\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_\mathfrak{n})+\alpha_2 D_b(\varsigma_{\mathfrak{n}+2},\varsigma_{\mathfrak{n}+1})+\alpha_3 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+1})\\ +\alpha_4 D_b(\varsigma_{\mathfrak{n}+2},\varsigma_\mathfrak{n})+\alpha_5 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\end{pmatrix}\right)$$

$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})+\alpha_1 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\\ +\alpha_4\omega[D_b(\varsigma_{\mathfrak{n}},\varsigma_{\mathfrak{n}+1})+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})]\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\alpha_5)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})+\alpha_1 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\\ +\alpha_4\omega D_b(\varsigma_{\mathfrak{n}},\varsigma_{\mathfrak{n}+1})+\alpha_4\omega D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\left(\phi\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\\ +(\alpha_1+\omega\alpha_4)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{pmatrix}\right)$$
$$\succcurlyeq \mathfrak{F}\begin{pmatrix}(\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\\ +(\alpha_1+\omega\alpha_4)\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{pmatrix}$$

The last inequality gives

$$(e-\alpha_1-\omega\alpha_4)\,\phi\big(D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\big)\succcurlyeq \mathfrak{F}\big((\alpha_2+\omega\alpha_4+\alpha_5)\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\big). \tag{3.2}$$

Put $(\alpha_2+\omega\alpha_4+\alpha_5)=\gamma$, then

$$(e-\alpha_1-\omega\alpha_4)\phi\big(D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\big)\succcurlyeq \mathfrak{F}\big(\gamma\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\big). \tag{3.3}$$

Since γ is invertible, to multiply γ^{-1} on both sides of (3.3)

$$\mathfrak{F}\big(\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\big)\preccurlyeq \mathfrak{h}\phi\big(D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\big)$$

Where $\mathfrak{h}=(e-\alpha_1-\omega\alpha_4)(\alpha_2+\omega\alpha_4+\alpha_5)^{-1}$. We get

$$\begin{aligned}\mathfrak{F}\big(\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}+2})\big)&\preccurlyeq \mathfrak{h}\phi\big(D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\big)\\ &\preccurlyeq \mathfrak{h}\big[\mathfrak{h}\phi\big(D_b(\varsigma_{\mathfrak{n}},\varsigma_{\mathfrak{n}-1})\big)\big]\\ &=\mathfrak{h}^2\phi\big(D_b(\varsigma_{\mathfrak{n}},\varsigma_{\mathfrak{n}-1})\big)\\ &\quad\vdots\\ &\preccurlyeq \mathfrak{h}^{\mathfrak{n}+1}\phi\big(D_b(\varsigma_1,\varsigma_0)\big).\end{aligned}$$

Now for $\mathfrak{m}\succ\mathfrak{n}$ we have by triangle inequality and $\rho(\mathfrak{h})\succ 1$,

$$\begin{aligned}\mathfrak{F}\big(\phi D_b(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{n}})\big)&\preccurlyeq \phi\big(D_b(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{m}-1})\big)+\phi\big(D_b(\varsigma_{\mathfrak{m}-1},\varsigma_{\mathfrak{n}})\big)+\cdots+\phi\big(D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\big)\\ &\preccurlyeq (\mathfrak{h}^{\mathfrak{m}-1}+\mathfrak{h}^{\mathfrak{m}-2}+\mathfrak{h}^{\mathfrak{m}-3}+\cdots+\mathfrak{h}^{\mathfrak{n}+1}+\mathfrak{h}^{\mathfrak{n}})\phi\big(D_b(\varsigma_1,\varsigma_0)\big)\\ &=(e+\mathfrak{h}^1+\mathfrak{h}^2+\mathfrak{h}^3+\cdots+\mathfrak{h}^{\mathfrak{m}-\mathfrak{n}-1})\phi\mathfrak{h}^{\mathfrak{n}}\big(D_b(\varsigma_1,\varsigma_0)\big)\\ &\preccurlyeq \left(\textstyle\sum_{i=0}^{\infty}\mathfrak{h}^i\right)\mathfrak{h}^{\mathfrak{n}}\phi\big(D_b(\varsigma_1,\varsigma_0)\big)\\ &\preccurlyeq \mathfrak{h}^{\mathfrak{n}}(e-\mathfrak{h})^{-1}\phi\big(D_b(\varsigma_1,\varsigma_0)\big)\end{aligned}$$

The properties of $\mathfrak{F}$ and ϕ implies that

$$D_b(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{n}})\preccurlyeq \mathfrak{h}^{\mathfrak{n}}(e-\mathfrak{h})^{-1}\big(D_b(\varsigma_1,\varsigma_0)\big)$$

By lemma 2.4 and the fact that $\|\mathfrak{h}^{\mathfrak{n}}(e-\mathfrak{h})^{-1}\big(D_b(\varsigma_1,\varsigma_0)\big)\|\to 0(\mathfrak{n}\to\infty)$ because of Remark 2.6, $\|\mathfrak{h}^{\mathfrak{n}}\|\to 0(\mathfrak{n}\to 0)$, it follows that $\forall\,\mathfrak{c}\in\mathfrak{A}$ with $\theta\ll\mathfrak{c},\exists\,\mathbb{N}\ni\forall\,\mathfrak{m}>n>N$, we have

$$D_b(\varsigma_{\mathfrak{m}},\varsigma_{\mathfrak{n}})\preccurlyeq \mathfrak{h}^{\mathfrak{n}}(e-\mathfrak{h})^{-1}\big(D_b(\varsigma_1,\varsigma_0)\big)\ll\mathfrak{c}.$$

Which implies that $\{\varsigma_{\mathfrak{n}}\}$ is Cauchy sequence. By the completeness of X, $\exists\,\varsigma^*\in\mathfrak{X}$, $\ni\varsigma_{\mathfrak{n}}\to\varsigma^*(\mathfrak{n}\to\infty)$. Consequently, we can find an $\varsigma^{**}\in\mathfrak{X}\ni\xi\varsigma^{**}=\varsigma^*$. Now we show that $\varsigma^*=\varsigma^{**}$. In fact,

$$\phi\big(D_b(\varsigma^*,\varsigma_{\mathfrak{n}})\big)=\phi\big(D_b(\xi\varsigma^{**},\xi\varsigma_{\mathfrak{n}+1})\big)$$

$$\phi\left(D_b(\xi\varsigma^{**},\xi\varsigma_{\mathfrak{n}+1})\right) \succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\frac{D_b(\varsigma^{**},\xi\varsigma^{**})}{1+D_b(\varsigma^{**},\xi\varsigma^{**})}+\alpha_2\frac{D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma_{\mathfrak{n}+1})}+\alpha_3\frac{D_b(\varsigma^{**},\xi\varsigma_{\mathfrak{n}+1})}{1+D_b(\varsigma^{**},\xi\varsigma_{\mathfrak{n}+1})}\\ +\alpha_4\frac{D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma^{**})}{1+D_b(\varsigma_{\mathfrak{n}+1},\xi\varsigma^{**})}+\alpha_5 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\end{array}\right)\right)$$

$$=\mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\frac{D_b(\varsigma^{**},\varsigma^{*})}{1+D_b(\varsigma^{**},\varsigma^{*})}+\alpha_2\frac{D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})}{1+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})}+\alpha_3\frac{D_b(\varsigma^{**},\varsigma_{\mathfrak{n}})}{1+D_b(\varsigma^{**},\varsigma_{\mathfrak{n}})}\\ +\alpha_4\frac{D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})}{1+D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})}+\alpha_5 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\end{array}\right)\right)$$

$$\succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1 D_b(\varsigma^{**},\varsigma^{*})+\alpha_2 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})+\alpha_3 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}})\\ +\alpha_4 D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})+\alpha_5 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\end{array}\right)\right)$$

Since the properties of $\mathfrak{F}$ and ϕ and triangle inequality, we have

$$D_b(\varsigma^{**},\varsigma^{*}) \succcurlyeq \omega[D_b(\varsigma^{*},\varsigma_{\mathfrak{n}+1})-D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})]$$
$$(\varsigma^{*},\varsigma_{\mathfrak{n}}) \preccurlyeq \omega[D_b(\varsigma^{*},\varsigma_{\mathfrak{n}+1})+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})]$$

and

$$D_b(\varsigma^{**},\varsigma_{\mathfrak{n}}) \preccurlyeq \omega[D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})]$$

It follows that

$$\phi\left(D_b(\varsigma^{*},\varsigma_{\mathfrak{n}})\right) \succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\omega[D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})]+\alpha_2 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\\ +\alpha_3\omega[D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})]\\ +\alpha_4 D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})+\alpha_5 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\end{array}\right)\right)$$

$$\succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\omega D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+\alpha_1\omega D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})+\alpha_2 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\\ +\alpha_3\omega D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+\alpha_3\omega D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\\ +\alpha_4 D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})+\alpha_5 D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\end{array}\right)\right)$$

$$\succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}(\alpha_1\omega+\alpha_3\omega+\alpha_5)D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+(\alpha_1\omega+\alpha_4)D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})\\ +(\alpha_2+\alpha_3\omega)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{array}\right)\right)$$

$$\phi\left(D_b(\varsigma^{*},\varsigma_{\mathfrak{n}})\right) \succcurlyeq \mathfrak{F}\left(\phi\left(\begin{array}{c}(\alpha_1\omega+\alpha_3\omega+\alpha_5)D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+(\alpha_1\omega+\alpha_4)D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})\\ +(\alpha_2+\alpha_3\omega)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{array}\right)\right)$$

The properties of $\mathfrak{F}$ and ϕ implies that

$$D_b(\varsigma^{*},\varsigma_{\mathfrak{n}}) \succcurlyeq \left(\begin{array}{c}(\alpha_1\omega+\alpha_3\omega+\alpha_5)D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})+(\alpha_1\omega+\alpha_4)D_b(\varsigma_{\mathfrak{n}+1},\varsigma^{*})\\ +(\alpha_2+\alpha_3\omega)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{array}\right)$$
$$(\alpha_1\omega+\alpha_3\omega+\alpha_5)D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1}) \preccurlyeq \left(\begin{array}{c}(e-\alpha_1\omega-\alpha_4)D_b(\varsigma^{*},\varsigma_{\mathfrak{n}+1})\\ +(e-\alpha_2+\alpha_3\omega)D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{array}\right)$$

Where

$$\mathfrak{F}\left(\phi D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\right) \preccurlyeq (\alpha_1\omega+\alpha_3\omega+\alpha_5)^{-1}\left[\begin{array}{c}(e-\alpha_1\omega-\alpha_4)\phi D_b(\varsigma^{*},\varsigma_{\mathfrak{n}+1})\\ +(e-\alpha_2+\alpha_3\omega)\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\end{array}\right].$$

This implies that

$$\mathfrak{F}(\phi D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})) \preccurlyeq \tau_1\phi D_b(\varsigma^{*},\varsigma_{\mathfrak{n}+1})+\tau_2\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})$$

where,

$\tau_1=(\alpha_1\omega+\alpha_3\omega+\alpha_5)^{-1}(e-\alpha_1\omega-\alpha_4)\in\mathfrak{A}$,
$\tau_2=(\alpha_1\omega+\alpha_3\omega+\alpha_5)^{-1}(e-\alpha_2+\alpha_3\omega)\in\mathfrak{A}$.

Now by proposition 2.3; $\{\tau_1\phi D_b(\varsigma^*,\varsigma_{\mathfrak{n}+1})+\tau_2\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\}$ are $\mathfrak{c}$-sequences and so $\{\tau_1 D_b(\varsigma^*,\varsigma_{\mathfrak{n}+1})+\tau_2 D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\}$ is also a $\mathfrak{c}$-sequences . Owing to $\varsigma_{\mathfrak{n}}\to\varsigma^*(\mathfrak{n}\to\infty)$, it follows by lemma 2.3. that $\forall\ \mathfrak{c}\in\mathfrak{A}$ with $\theta\ll\mathfrak{c},\exists\ \mathfrak{N}\in\mathbb{N}$ such that

$$\mathfrak{F}\left(\phi D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\right)\preccurlyeq\tau_1\phi D_b(\varsigma^*,\varsigma_{\mathfrak{n}+1})+\tau_2\phi D_b(\varsigma_{\mathfrak{n}+1},\varsigma_{\mathfrak{n}})\ll\mathfrak{c},\qquad\forall\,\mathfrak{n}>N$$

Hence $\mathfrak{F}\left(\phi D_b(\varsigma^{**},\varsigma_{\mathfrak{n}+1})\right)\ll\mathfrak{c}$. Since the limit of a convergent sequence in cone b-metric space over Banach algebra is unique, we have $\varsigma^*=\varsigma^{**}, i.e., \varsigma^*$ is a fixed point of ξ.

Finally, we prove the uniqueness of the fixed point. In fact, if ϱ^* is another common fixed point of ξ, that is, $\xi\varrho^*=\varrho^*$, form (3.1) we have

$$\phi(D_b(\xi\varsigma^*,\xi\varrho^*))\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma^*,\xi\varsigma^*)}{1+D_b(\varsigma^*,\xi\varsigma^*)}+\alpha_2\frac{D_b(\varrho^*,\xi\varrho^*)}{1+D_b(\varrho^*,\xi\varrho^*)}+\alpha_3\frac{D_b(\varsigma^*,\xi\varrho^*)}{1+D_b(\varsigma^*,\xi\varrho^*)}\\+\alpha_4\frac{D_b(\varrho^*,\xi\varsigma^*)}{1+D_b(\varrho^*,\xi\varsigma^*)}+\alpha_5 D_b(\varsigma^*,\varrho^*)\end{pmatrix}\right)$$

$$=\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma^*,\varsigma^*)}{1+D_b(\varsigma^*,\varsigma^*)}+\alpha_2\frac{D_b(\varrho^*,\varrho^*)}{1+D_b(\varrho^*,\varrho^*)}+\alpha_3\frac{D_b(\varsigma^*,\varrho^*)}{1+D_b(\varsigma^*,\varrho^*)}\\+\alpha_4\frac{D_b(\varrho^*,\varsigma^*)}{1+D_b(\varrho,\varsigma^*)}+\alpha_5 D_b(\varsigma^*,\varrho^*)\end{pmatrix}\right)$$

$$\phi(D_b(\varsigma^*,\varrho^*))\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1 D_b(\varsigma^*,\varsigma^*)+\alpha_2 D_b(\varrho^*,\varrho^*)+\alpha_3 D_b(\varsigma^*,\varrho^*)\\+\alpha_4 D_b(\varrho^*,\varsigma^*)+\alpha_5 D_b(\varsigma^*,\varrho^*)\end{pmatrix}\right)$$

The last inequality gives

$$(\alpha_3+\alpha_4+\alpha_5-e)\mathfrak{F}\left(\phi D_b(\varsigma^*,\varrho^*)\right)\preccurlyeq\theta$$

Thus, $\mathfrak{F}(\theta)=\theta$, then $\mathfrak{F}\left(\phi D_b(\varsigma^*,\varrho^*)\right)=\theta$. This implies that $D_b(\varsigma^*,\varrho^*)=\theta$. It follows that $\varsigma^*=\varrho^*$, a contradiction show that the fixed point must be unique.

Corollary 3.3

Let $(\mathfrak{X},D_b)$ be a complete cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be an underlying solid cone in $\mathfrak{A}$ with the coefficient $\omega\succcurlyeq 1$. Let mapping $\xi:\mathfrak{X}\to\mathfrak{X}$ be a surjective self-map of $\mathfrak{X}$ satisfies the expansive condition

$$\phi(D_b(\xi\varsigma,\xi\varrho))\succcurlyeq\mathfrak{F}(\phi(\mathcal{M}(\varsigma,\varrho))$$

where,

$$\mathcal{M}(\varsigma,\varrho)=\alpha_1\frac{D_b(\varsigma,\xi\varsigma)}{1+D_b(\varsigma,\xi\varsigma)}+\alpha_2\frac{D_b(\varrho,\xi\varrho)}{1+D_b(\varrho,\xi\varrho)}+\alpha_3\frac{D_b(\varrho,\xi\varsigma)}{1+D_b(\varrho,\xi\varsigma)}+\alpha_4 D_b(\varsigma,\varrho)$$

for all $\varsigma,\varrho\in\mathfrak{X}$, where $\alpha_i\in\mathfrak{P}$, $(i=1,2,3,4)$ such that $(\alpha_2+\omega\alpha_3+\alpha_5)^{-1}$, $(\alpha_1\omega+\alpha_4)^{-1}\in\mathfrak{P}$ and spectral radius $\rho\left((e-\alpha_1-\omega\alpha_3)(\alpha_2+\omega\alpha_3+\alpha_4)^{-1}\right)\prec 1$. Then ξ has a unique fixed point.

Corollary 3.4

Let $(\mathfrak{X},D_b)$ be a complete cone b-metric space over Banach algebra $\mathfrak{A}$ and $\mathfrak{P}$ be an underlying solid cone in $\mathfrak{A}$ with the coefficient $\omega\succcurlyeq 1$. Let mapping $\xi:\mathfrak{X}\to\mathfrak{X}$ be a surjective self-map of $\mathfrak{X}$ satisfies the expansive condition

$$\phi(D_b(\xi\varsigma,\xi\varrho))\succcurlyeq\mathfrak{F}(\phi(\mathcal{M}(\varsigma,\varrho))$$

where,

$$\mathcal{M}(\varsigma,\varrho)=\alpha_1 D_b(\varsigma,\xi\varrho)+\alpha_2 D_b(\varrho,\xi\varsigma)+\alpha_3 D_b(\varsigma,\varrho)$$

for all $\varsigma,\varrho\in\mathfrak{X}$, where $\alpha_i\in\mathfrak{P}$, $(i=1,2,3,4,5)$ such that $(\omega\alpha_2+\alpha_3)^{-1},(\alpha_1\omega+\alpha_3)^{-1}\in\mathfrak{P}$ and spectral radius $\rho\left((e-\omega\alpha_2)(\omega\alpha_2+\alpha_3)^{-1}\right)\prec 1$. Then ξ has a unique fixed point.

Example 3.5

Let the Banach algebra $\mathfrak{A}$ and the cone $\mathfrak{P}$ be the same ones as those in example 2.8. Let $\mathfrak{X}=\left\{0,\frac{1}{2},\frac{3}{2}\right\}$, and let $D_b:\mathfrak{X}\times\mathfrak{X}\to\mathfrak{A}$. Clearly, $(\mathfrak{X},D_b)$ is a cone b-metric space over Banach algebra $\mathfrak{A}$ with coefficient $\omega=\frac{4}{3}\succ 1$. Define $\mathfrak{F}:\mathfrak{P}\to\mathfrak{P}$ by $\mathfrak{F}(\mathrm{t})\prec\mathrm{t}$ for all $\mathrm{t}\succ 0$. Then $\mathfrak{F}\in\Psi$. Also define $\phi:\mathfrak{P}\to\mathfrak{P}$ by $\phi(\mathrm{t})=\mathcal{K}\mathrm{t}$ for some $\mathcal{K}\succ 0$. Then ϕ is a continuous comparison function. Now define the mapping $\xi:\mathfrak{X}\to\mathfrak{X}$ by

1) $D_b(\varsigma,\varrho)=0, where\ \varsigma=\varrho,\forall\ \varsigma,\varrho\in\mathfrak{X}$.
2) $D_b\left(0,\frac{1}{2}\right)=D_b\left(\frac{1}{2},0\right)=\frac{1}{4},D_b\left(0,\frac{3}{2}\right)=D_b\left(\frac{3}{2},0\right)=\frac{1}{8},D_b\left(\frac{1}{2},\frac{3}{2}\right)=D_b\left(\frac{3}{2},\frac{1}{2}\right)=\frac{1}{2}$.

Define by $\xi(0)=0,\xi\left(\frac{1}{2}\right)=\frac{3}{2},\xi\left(\frac{3}{2}\right)=0$. Let $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\frac{1}{6}$, clearly, $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5\prec 1$. Next we will verify the condition (3.1). It have the following cases to be considered.

Case (i): $\phi\left(D_b(\xi\varsigma,\xi\varrho)\right)=0$, the inequality (3.1) holds.
Case (ii): $\phi\left(D_b(\xi\varsigma,\xi\varrho)\right)\neq 0$, we have following cases to be considered.
Case (ii-a): $\varsigma=0,\varrho=\frac{1}{2}$, we can get $\phi\left(D_b(\xi\varsigma,\xi\varrho)\right)=\frac{1}{2}$, then

$$\frac{1}{2}\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\frac{1}{6}\frac{D_b(0,\xi(0))}{1+D_b(0,\xi(0))}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(0,\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(0,\xi\left(\frac{1}{2}\right)\right)}\\+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi(0)\right)}{1+D_b\left(\frac{1}{2},\xi(0)\right)}+\frac{1}{6}D_b\left(0,\frac{1}{2}\right)\end{pmatrix}\right)$$

$$\succcurlyeq\frac{1}{6}\begin{pmatrix}\frac{D_b(0,0)}{1+D_b(0,0)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(0,\frac{3}{2}\right)}{1+D_b\left(0,\frac{3}{2}\right)}\\+\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+D_b\left(0,\frac{1}{2}\right)\end{pmatrix}$$

$$\succcurlyeq\frac{1}{6}\left(0+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{8}}{1+\frac{1}{8}}+\frac{\frac{1}{4}}{1+\frac{1}{4}}+\frac{1}{4}\right)$$

$$\succcurlyeq\frac{1}{6}\left(\frac{1}{2}\times\frac{2}{3}+\frac{1}{8}\times\frac{8}{9}+\frac{1}{4}\times\frac{4}{5}+\frac{1}{4}\right)$$

$$\succcurlyeq\frac{1}{6}\left(\frac{1}{3}+\frac{1}{9}+\frac{1}{5}+\frac{1}{4}\right)$$

$$\frac{1}{2}\succcurlyeq\frac{1}{6}\left(\frac{161}{180}\right)$$

$$\phi\left(D_b(\xi\varsigma,\xi\varrho)\right)\succcurlyeq\mathfrak{F}\left(\phi\frac{1}{6}\left(\mathcal{M}(\varsigma,\varrho)\right)\right)$$

$$\varphi\left(D_b(\xi\varsigma,\xi\varrho)\right)\succcurlyeq\mathfrak{F}\left(\phi\begin{pmatrix}\alpha_1\frac{D_b(\varsigma,\xi\varsigma)}{1+D_b(\varsigma,\xi\varsigma)}+\alpha_2\frac{D_b(\varrho,\xi\varrho)}{1+D_b(\varrho,\xi\varrho)}+\alpha_3\frac{D_b(\varsigma,\xi\varrho)}{1+D_b(\varsigma,\xi\varrho)}\\+\alpha_4\frac{D_b(\varrho,\xi\varsigma)}{1+D_b(\varrho,\xi\varsigma)}+\alpha_5 D_b(\varsigma,\varrho)\end{pmatrix}\right)$$

Therefore, the inequality (3.1) holds.

Case (ii-b): $\varsigma=\frac{1}{2},\varrho=\frac{1}{2}$, we can get $\phi\left(D_b(\xi\varsigma,\xi\varrho)\right)=\frac{1}{2}$, then

$$\frac{1}{2} \geqslant \mathfrak{F}\left(\phi\left(\begin{array}{c}\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}\\+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}+\frac{1}{6}D_b\left(\frac{1}{2},\frac{1}{2}\right)\end{array}\right)\right)$$

$$\geqslant \frac{1}{6}\left(\begin{array}{c}\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}\\+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+D_b\left(\frac{1}{2},\frac{1}{2}\right)\end{array}\right)$$

$$\geqslant \frac{1}{6}\left(\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+0\right)$$

$$\geqslant \frac{1}{6}\left(\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{2}\times\frac{2}{3}\right)$$

$$\geqslant \frac{1}{6}\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$$

$$\frac{1}{2} \geqslant \frac{1}{6}\left(\frac{4}{3}\right)$$

$$\phi\big(D_b(\xi\varsigma,\xi\varrho)\big) \geqslant \mathfrak{F}\left(\phi\,\frac{1}{6}\big(\mathcal{M}(\varsigma,\varrho)\big)\right)$$

$$\varphi\big(D_b(\xi\varsigma,\xi\varrho)\big) \geqslant \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\frac{D_b(\varsigma,\xi\varsigma)}{1+D_b(\varsigma,\xi\varsigma)}+\alpha_2\frac{D_b(\varrho,\xi\varrho)}{1+D_b(\varrho,\xi\varrho)}+\alpha_3\frac{D_b(\varsigma,\xi\varrho)}{1+D_b(\varsigma,\xi\varrho)}\\+\alpha_4\frac{D_b(\varrho,\xi\varsigma)}{1+D_b(\varrho,\xi\varsigma)}+\alpha_5 D_b(\varsigma,\varrho)\end{array}\right)\right)$$

Therefore, the inequality (3.1) holds.

Case (ii-c): $\varsigma = \frac{3}{2}, \varrho = \frac{1}{2}$, we can get $\phi\big(D_b(\xi\varsigma,\xi\varrho)\big) = \frac{1}{2}$, then

$$\frac{1}{2} \geqslant \mathfrak{F}\left(\phi\left(\begin{array}{c}\frac{1}{6}\frac{D_b\left(\frac{3}{2},\xi\left(\frac{3}{2}\right)\right)}{1+D_b\left(\frac{3}{2},\xi\left(\frac{3}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{1}{2}\right)\right)}+\frac{1}{6}\frac{D_b\left(\frac{3}{2},\xi\left(\frac{1}{2}\right)\right)}{1+D_b\left(\frac{3}{2},\xi\left(\frac{1}{2}\right)\right)}\\+\frac{1}{6}\frac{D_b\left(\frac{1}{2},\xi\left(\frac{3}{2}\right)\right)}{1+D_b\left(\frac{1}{2},\xi\left(\frac{3}{2}\right)\right)}+\frac{1}{6}D_b\left(\frac{3}{2},\frac{1}{2}\right)\end{array}\right)\right)$$

$$\geqslant \frac{1}{6}\left(\begin{array}{c}\frac{D_b\left(\frac{3}{2},0\right)}{1+D_b\left(\frac{3}{2},0\right)}+\frac{D_b\left(\frac{1}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{1}{2},\frac{3}{2}\right)}+\frac{D_b\left(\frac{3}{2},\frac{3}{2}\right)}{1+D_b\left(\frac{3}{2},\frac{3}{2}\right)}\\+\frac{D_b\left(\frac{1}{2},0\right)}{1+D_b\left(\frac{1}{2},0\right)}+D_b\left(\frac{3}{2},\frac{1}{2}\right)\end{array}\right)$$

$$\geqslant \frac{1}{6}\left(\frac{\frac{1}{8}}{1+\frac{1}{8}}+\frac{\frac{1}{2}}{1+\frac{1}{2}}+0+\frac{\frac{1}{4}}{1+\frac{1}{4}}+\frac{1}{2}\right)$$

$$\geqslant \frac{1}{6}\left(\frac{1}{8}\times\frac{8}{9}+\frac{1}{2}\times\frac{2}{3}+\frac{1}{4}\times\frac{4}{5}+\frac{1}{2}\right)$$

$$\geqslant \frac{1}{6}\left(\frac{1}{9}+\frac{1}{3}+\frac{1}{5}+\frac{1}{2}\right)$$

$$\frac{1}{2} \geqslant \frac{1}{6}\left(\frac{103}{90}\right)$$

$$\phi\big(D_b(\xi\varsigma,\xi\varrho)\big) \geqslant \mathfrak{F}\left(\phi\,\frac{1}{6}\big(\mathcal{M}(\varsigma,\varrho)\big)\right)$$

$$\phi\big(D_b(\xi\varsigma,\xi\varrho)\big) \geqslant \mathfrak{F}\left(\phi\left(\begin{array}{c}\alpha_1\frac{D_b(\varsigma,\xi\varsigma)}{1+D_b(\varsigma,\xi\varsigma)}+\alpha_2\frac{D_b(\varrho,\xi\varrho)}{1+D_b(\varrho,\xi\varrho)}+\alpha_3\frac{D_b(\varsigma,\xi\varrho)}{1+D_b(\varsigma,\xi\varrho)}\\+\alpha_4\frac{D_b(\varrho,\xi\varsigma)}{1+D_b(\varrho,\xi\varsigma)}+\alpha_5 D_b(\varsigma,\varrho)\end{array}\right)\right)$$

Therefore, the inequality (3.1) holds. Where $\alpha_i \in \mathfrak{P}$, $(i = 1,2,3,4,5)$ such that

$$(\alpha_2+\omega\alpha_4+\alpha_5)^{-1}, (\alpha_1\omega+\alpha_3\omega+\alpha_5)^{-1} \in \mathfrak{P}$$

and spectral radius

$$\rho\big((e-\alpha_1-\omega\alpha_4)(\alpha_2+\omega\alpha_4+\alpha_5)^{-1}\big) \prec 1$$

Therefore, we showed that the condition (3.1) is satisfied in all cases. Then ξ has a unique fixed point $\varsigma = 0 \in \mathfrak{X}$.

4 Conclusions

In Theorem 3.2 we have formulated a new contractive conditions to modify and extend some fixed point theorem $(\phi, \mathfrak{F})$- expansive mapping in cone b-metric space over Banach algebra. The existence and uniqueness of the result is presented in this article. We have also given some example which satisfies the condition of our main result. Our result may be the vision for other authors to extend and improve several results in such spaces and applications to other related areas.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Czerwik S. Contraction mappings in b-metric spaces. Acta Mathematica et Informatica Universitatis Ostraviensis. 1993;1:5–11.

[2] Czerwik S. Nonlinear set-valued contraction mappings in b-metric spaces. Atti del Seminario Matematico e Fisico Universita di Modena. 1998;46(2):263–276.

[3] Wang SZ, Li BY, Gao ZM, Iseki K. Some fixed point theorems for expansion mapping, Mathematica Japonica. 1984;29:631-636.

[4] Aage CT, Salunke JN. Some fixed point theorems for expansion onto mappings on cone metric space, Acta Mathematica Sinica English Series. 2011;27(6):1101-1106.

[5] Daffer PZ, Kaneko H. On expansive mappings. Mathematica Japonica. 1992;37:733-735.

[6] Kadelburg Z, Radenovic S. A note on various types of cones and fixed point results in cone metric space. Asian Journal Mathematics Applications; 2013.

[7] Dhamodharan D, and Krishnakumar R. Common fixed points of $(\boldsymbol{\psi}, \boldsymbol{\varphi})$-weak contractions in regular cone metric spaces. International Journal of Mathematical Archive. 2017;8(8):1-8.

[8] Nagoor Gani A, Mohamed Althaf M. Fixed Point theorems in strong fuzzy metric spaces using control function. International Journal of Pure and Applied Mathematics. 2018;118(6):389-397.

[9] Nagoor Gani A, Mohamed Althaf M. Fixed point theorems in strong fuzzy metric spaces using generalized altering distance function. Journal of Physical Sciences. 2018;23:67-75.

[10] Kadelburg Z, Pavlovic M, Radenovic S. Common fixed point theorems for ordered contractions and quasi-contractions in orderd cone metric spaces. Computers and Mathematics with Applications. 2010;59:3148-3159.

[11] Radenovic S, Rhoades BE. Fixed point theorem for two non-self-mappings in cone metric spaces. Computers and Mathematics with Applications. 2009;57:1701-1707.

[12] Rudin W. Functional analysis. MeGraw-Hill, New York; 1991.

[13] Jiang B, Xu S, Huang H, Cai Z. Some fixed point theorems for generalized expansive mappings in cone metric space over Banach algebras. Journal of Computational Analysis and Applications. 2016; 21(6):1103-1114.

[14] Xu S, Radenovic S. Fixed point theorems of generalized Lipschitz mappings on cone metric spaces over Banach algebra without assumption of normality. Fixed Point Theory and Applications. 2014;12.

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