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An Optimal Class of Fourth-order Iterative Methods without Restraint on the First Derivative

Malak M. Khashoqji [a] and I. A. Al-Subaihi [b*]

[a] *Department of Electrical Engineering, University of Prince Mugrin, Saudi Arabia.*
[b] *Department of General Studies, University of Prince Mugrin, Saudi Arabia.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

In an attempt to create an iterative method that may converge even if the first derivative disappears during the recursive process. This paper sets out to develop a class of optimal fourth-order methods based on Wu's modified Newton scheme for solving nonlinear equations without constraints on the first derivative. Numerous numerical examples were given to demonstrate how effectively the proposed methods perform. In addition, the basins of attraction confirm the efficiency and performance of the suggested fourth-order method compared with some other fourth-order schemes.

Keywords: Newton's method; iterative methods; weight function; efficiency index; order of convergence.

2010 Mathematics Subject Classification: 65H05, 65B99.

Corresponding author: E-mail: alsubaihi@hotmail.com;

1 Introduction

The most common and early issue in innumerable scientific and engineering fields involves solving a nonlinear system of equations, see [1], and its references. Throughout this paper, we will discuss the solution of nonlinear equations. While traditional approaches are unable to solve problems of this kind, a diversity of iterative approaches have been developed to reach a simple zero of the function $f(x) = 0$ where, $f : I \subseteq \mathbb{R} \to \mathbb{R}$ for an open interval I. A well-known approach for solving nonlinear equations and the most widely used to $f(x) = 0$ is Newton's method (NM), see [2–18] and their references. The conventional Newton's scheme can be written as:

$$x_{\delta+1} = \frac{f(x_\delta)}{f'(x_\delta)}, \quad \delta = 0, 1, 2, 3, \dots . \tag{1.1}$$

The efficiency index (EI) can be described by $\sqrt[\tau]{\rho}$ where, ρ represents the order of the method and τ is the total number of times the functions and their derivatives have been evaluated [19]. A second-order rate of convergence is achieved by Newton's method [3], with an efficiency index of $\sqrt{2} \approx 1.4142$. According to the Kung-Traub conjecture [21] which is given by $2^{\tau-1}$, wherein an iterative process (NM) is an optimal method. Various modifications have been made to Newton's scheme to increase its convergence rate [2–13, 17–25]. However, the majority of these methods overlooked the fact that during the iterative process, the first derivative disappears. The preceding alteration to the traditional Newton's technique was presented by Wu [18].

$$x_{\delta+1} = \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)}, \quad \theta = sign(f(x_\delta) f'(x_\delta)). \tag{1.2}$$

Therefore, the denominator is never zero regardless of whether the first derivative disappears or not. Recently a sixteenth-order recursive scheme has been developed by Khattri et al (KAM) [12] based on Wu's method. The efficiency index for (KAM) is $\sqrt[6]{16} \approx 1.5874$.

$$\begin{aligned}
y_\delta &= x_\delta - \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)}, \\
z_\delta &= y_\delta - \frac{f(y_\delta)}{f'(x_\delta) + \theta f(x_\delta)} \Phi(x_\delta), \\
x_{\delta+1} &= z_\delta - \frac{f(z_\delta)}{f'(z_\delta) + \beta f(z_\delta)} - \frac{f(\frac{f(z_\delta)}{f'(z_\delta) + \beta f(z_\delta)})}{f'(z_\delta) + \beta f(z_\delta)} \zeta(x_\delta),
\end{aligned} \tag{1.3}$$

where the functions $\Phi(x_\delta)$ and $\zeta(x_\delta)$ are given as:

$$\begin{aligned}
\Phi(x_\delta) &= 1 + 2\frac{\frac{f(x_\delta - f(x_\delta))}{f'(x_\delta)} + \theta f(x_\delta)}{f(x_\delta)} - \theta \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)} \\
&\quad + 5(\frac{\frac{f(x_\delta - f(x_\delta))}{f'(x_\delta)}}{f(x_\delta)})^2 + (\theta \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)}))^2 \\
&\quad - 5\theta(\frac{\frac{f(x_\delta - f(x_\delta))}{f'(x_\delta)} + \theta f(x_\delta)}{f(x_\delta)})(\frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)}), \\
\zeta(x_\delta) &= 1 + 2\frac{\frac{f(z_\delta - f(z_\delta))}{f'(z_\delta)} + \beta f(z_\delta)}{f(z_\delta)} - \beta \frac{f(z_\delta)}{f'(z_\delta) + \beta f(z_\delta)} \\
&\quad + 5(\frac{\frac{f(z_\delta - f(z_\delta))}{f'(z_\delta)}}{f(z_\delta)})^2 + (\theta \frac{f(z_\delta)}{f'(z_\delta) + \beta f(z_\delta)}))^2 \\
&\quad - 5\beta(\frac{\frac{f(z_\delta - f(z_\delta))}{f'(z_\delta)} + \beta f(z_\delta)}{f(z_\delta)})(\frac{f(z_\delta)}{f'(z_\delta) + \beta f(z_\delta)}).
\end{aligned} \tag{1.4}$$

Where, $\theta = sign(f(x_\delta) f'(x_\delta))$, and $\beta = sign(f(z_\delta) f'(z_\delta))$, so that none of the denominators of method (1.3) is zero. While the proposed method is quite effective, however, it is not optimal. As it does not satisfy the

Kung-Traub conjecture. To elaborate, in order to achieve optimality the rate thirty second must be converged to by (KAM) with its current total number of function evaluations.

2 The Method

This paper aims to improve KAM (1.3) by constructing a class of optimal fourth-order iterative algorithms. The proposed method converges regardless of whether the first derivative disappears during the iteration process or not, by combining two methods and employing a weight function.

Theorem 2.1 ([24]). *Let $\omega_1(x)$,$\omega_2(x)$,...,$\omega_\delta(x)$ be the iterative algorithms with the convergence rate λ_1,λ_2,...,λ_δ consecutively, as a result, the iterative algorithms are composed to be $\omega_1(\omega_2(...(\omega_\delta(x))))$ which defines the convergence rate to be the product of their orders $\lambda_1\lambda_2...\lambda_\delta$.*

Applying theorem 2.1, followed by Wu's scheme as a second step we obtain the following algorithm:

$$\begin{aligned} y_\delta &= x_\delta - \frac{f(x_\delta)}{f'(x_\delta)+\theta f(x_\delta)}, \\ x_{\delta+1} &= y_\delta - \frac{f(y_\delta)}{f'(y_\delta)+\theta f(y_\delta)}. \end{aligned} \tag{2.1}$$

Upon satisfying the Kung-Traub conjecture, four functions need to be evaluated. Hence, it is necessary to approximate $f'(y_\delta)$ using Chun et al approximation [7] in order for the preceding algorithm to be optimal.

$$q'(y_\delta) \approx \frac{f'(x_\delta)f(x_\delta)^2}{(f(x_\delta)+f(y_\delta))^2}. \tag{2.2}$$

Minimizing the number of function evaluations from $\tau = 4$ to $\tau = 3$. Additionally, adding a real-valued weight function $H(v)$ to $q'(y_\delta)$, where $v = f(x_\delta)$. Aiming to reach a class of optimal four, with an EI of $\sqrt[3]{4} \approx 1.5874$.

$$\begin{aligned} y_\delta &= x_\delta - \frac{f(x_\delta)}{f'(x_\delta)+\theta f(x_\delta)}, \\ x_{\delta+1} &= y_\delta - \frac{f(y_\delta)}{[q'(y_\delta)+H(v)]+\theta f(y_\delta)}. \end{aligned} \tag{2.3}$$

The following theorem 2.2 demonstrates the constraints of the weight function. Verifying the equation's optimality toward the fourth convergence rate.

Theorem 2.2. *Let $\sigma \in I$ be the simple zero of an effectively differentiable function $f : I \subseteq \mathbb{R} \to \mathbb{R}$. Under the following conditions, if $x_\circ$ is fairly close to σ, and $H(0) = 0,\ H'(0) = 2,\ |H''(0)| < \infty$. Therefore, method (2.3) yields toward optimal four.*

The following results are obtained using Maple 2022 to verify the convergence rate of the proposed method namely (KSM).

Proof. Let σ be a simple zero of the function, and let $e_\delta = x_\delta - \sigma$ to be the error at the δ^{th} iteration. Using Taylor expansion, the following is obtained: ■

$$f(x_\delta) = f'(\sigma)\left[e + c_2e^2 + c_3e^3 + c_4e^4 + O(e^5)\right], \tag{2.4}$$

$$f'(x_\delta) = f'(\sigma)\left[1 + 2c_2e + 3c_3e^2 + 4c_4e^3 + 5c_5e^4 + O(e^5)\right], \tag{2.5}$$

where $C_j = \frac{f^{(j)}(\sigma)}{j!f'(\sigma)}$, $j = 2, 3, \ldots$.

Adding (2.4) and (2.5), where θ represents the sign to $f(x_\delta)f'(x_\delta)$

$$f'(x_\delta) + \theta f(x_\delta) = f'(\sigma)\left[1 + (1 + 2c_2)e + \ldots + (c_4 + 5c_5)e^4 + O(e^5)\right]\cdot \tag{2.6}$$

Furthermore, dividing (2.4) by (2.6) to get:

$$\begin{aligned}\frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)} &= e + (-c_2 - 1)e^2 + (2c_2^2 + 2c_2 - 2c_3 + 1)e^3 \\ &+ (-4c_2^3 - 5c_2^2 + 7c_2c_3 - 3c_2 + 4c_3 - 3c_4 - 1)e^4 + O(e^5)\cdot\end{aligned} \tag{2.7}$$

Substituting equation (2.7) into the first step of (2.3), to obtain:

$$\begin{aligned}y_\delta &= \sigma + (c_2 + 1)e^2 + (-2c_2^2 - 2c_2 + 2c_3 - 1)e^3 \\ &+ (4c_2^3 + 5c_2^2 - 7c_2c_3 + 3c_2 - 4c_3 + 3c_4 + 1)e^4 + O(e^5)\cdot\end{aligned} \tag{2.8}$$

Expanding $f(y_\delta)$ about σ using Taylor expansion to get:

$$\begin{aligned}f(y_\delta) &= f'(\sigma)\left[(c_2)e^2 + (-2c_2^2 - 2c_2 + 2c_3 - 1)e^3 \right.\\ &\left.+ \left(5c_2^3 + 7c_2^2 - 7c_2c_3 + 4c_2 - 4c_3 + 3c_4 + 1\right)e^4 + O(e^5)\right]\cdot\end{aligned} \tag{2.9}$$

From (2.4), (2.5), and (2.9) the approximation can be easily calculated as follow:

$$\begin{aligned}q'(y_\delta) &= f'(\sigma)\left[1 - 2e + ... - 103c_3c_2^2 + 26c_2c_4 + 233c_2^2 + 124c_2 - ... \right.\\ &\left.- \left(3c_5 + ... - 154c_2 + 214c_2^3 + 28\right)e^4 + O(e^5)\right]\cdot\end{aligned} \tag{2.10}$$

Using (2.9) and (2.10), to get the following:

$$\begin{aligned}q'(y_\delta) + \theta f(y_\delta) &= f'(\sigma)\left[1 - 2e + (5c_2^2 - c_3 + 6 + 9c_2)e^2 + (18c_2c_3 - ... - 38c_2)e^3 \right.\\ &\left.+ \left(-103c_3c_2^2 + 26c_2c_4 - ... + 27c_4 - 3c_5 + 240c_2^2\right)e^4 + +O(e^5)\right]\cdot\end{aligned} \tag{2.11}$$

Hence, dividing (2.9) and (2.11) to obtain:

$$\begin{aligned}\frac{f(y_\delta)}{q'(y_\delta) + \theta f(y_\delta)} &= f'(\sigma)\left[(c_2 + 1)e^2 + (-2c_2^2 + 2c_3 + 1)e^3 \right.\\ &\left.+ \left(-11c_2^2 - 6c_2c_3 - 11c_2 + c_3 + 3c_4 - 3\right)e^4 + O(e^5)\right]\cdot\end{aligned} \tag{2.12}$$

Using Taylor's polynomial of the fourth order at v = 0 to expand H(v):

$$H(v) = H(0) + H(0)^{(1)}v + H(0)^{(2)}\frac{v^2}{2!} + H(0)^{(3)}\frac{v^3}{3!} + H(0)^{(4)}\frac{v^4}{4!}, \quad v = f(x)\cdot \tag{2.13}$$

Then, substituting (2.12) into the second step of (2.3), and adding (2.13) to the denominator using the conditions in theorem 2.2. As a result, the following error expression is received:

$$x_{\delta+1} = \sigma + (6 - c_3 + 16c_2 + 14c_2^2 + 4c_2^3 - c_2c_3 + f'(\sigma)\frac{H''}{2}c_2 + f'(\sigma)\frac{H''}{2})e^4 + O(e^5)\cdot \tag{2.14}$$

Accordingly, the class in (2.3) has an exact order of convergence of four and as a result, the proof is complete.

In particular, the following cases fall under the class of (2.3), where $\theta = sign(f(x_\delta)f'(x_\delta))$, and $v = f(x)$.
For $H = 2v$ a new fourth-order method is established, namely (KSM1):

$$y_\delta = x_\delta - \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{f(y_\delta)}{[q'(y_\delta) + 2v] + \theta f(y_\delta)}. \tag{2.15}$$

For $H = 2\sin(v)$, a development of another fourth-order method is acquired under the name (KSM2):

$$y_\delta = x_\delta - \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{f(y_\delta)}{[q'(y_\delta) + 2\sin(v)] + \theta f(y_\delta)}. \tag{2.16}$$

Repeating the above process, with the value of $H = 3v^2 + 2v$, also lead to another fourth-order method known as (KSM3):

$$y_\delta = x_\delta - \frac{f(x_\delta)}{f'(x_\delta) + \theta f(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{f(y_\delta)}{[q'(y_\delta) + 3v^2 + 2v] + \theta f(y_\delta)}. \tag{2.17}$$

3 Numerical Example

This part outlines several numerical tests along with the basins of attraction to indicate the efficiency and performance of the new suggested technique (KSM) compared to other three optimal fourth-order methods, such as:
King Method (KM) [20], where $\beta = 3$:

$$y_\delta = x_\delta - \frac{f(x_\delta)}{f'(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{f(y_\delta)}{f'(x_\delta)} \frac{f(x_\delta) + \beta f(y_\delta)}{f(x_\delta) + (\beta - 2) f(y_\delta)}. \tag{3.1}$$

Furthermore Hafiz et al also developed the following optimal fourth-order method known as (HKM) [9], where a=2, b=4 and, c=-5:

$$y_\delta = x_\delta - \frac{2f(x_\delta)}{3f'(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{f(x_\delta)}{f'(x_\delta) + f'(y_\delta)W(\eta)},$$
$$where \;\; W(\eta) = \frac{1}{6}\left[a + b\eta + cln(2 - \eta) \geq 1\right], \;\; \eta = \frac{f'(x_\delta)}{f'(y_\delta)}. \tag{3.2}$$

In addition, Laila created the following scheme (LM) [2], where $B = -2$, and $a = 2$:

$$y_\delta = x_\delta - \frac{f(x_\delta)}{f'(x_\delta)},$$
$$x_{\delta+1} = y_\delta - \frac{4(y_\delta - x_\delta) f(y_\delta)}{(y_\delta - x_\delta)\left[f'(x_\delta) + f_1'(y_\delta)\right] + 2\left[f(y_\delta) - f(x_\delta)\right]} H(v),$$

$$where \;\; f_1'(y_\delta) \approx \frac{f'(x_\delta) f'(x_\delta)^2}{(f(x_\delta) + f(y_\delta))^2}, \;\; and \;\; H(v) = 1 + v + Bv^a, \;\; v = \frac{f(y_\delta)}{f(x_\delta)}. \tag{3.3}$$

The test functions taken into consideration are listed in Table 1, to test the newly proposed class toward the optimal fourth-order convergence rate along with their simple root σ, and their preliminary assumptions $x_\circ$.

Table 1. Test functions, their roots, and their preliminary assumptions

Function	Preliminary Assumption	Root
$f(x)_1 = \sin(x) - \frac{x}{2}$	$x_\circ = 2$	$\sigma = 1.895494267033980$
$f(x)_2 = e^{-x^2+x+2} - 1$	$x_\circ = 2.5$	$\sigma = 2.0$
$f(x)_3 = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$	$x_\circ = 1.5$	$\sigma = -1.0$
$f(x)_4 = (x-1)^4 - 1$	$x_\circ = 1.5$	$\sigma = 2.0$
$f(x)_5 = \sin(2\cos(x)) - 1 - (x^2) + e^{\sin(x^3)}$	$x_\circ = 1.2$	$\sigma = 1.306175201846827$

With a 1000-digit floating point, all calculations were carried out in MATLAB (R2022a). Table 2 lists the iterations number, the function's absolute value $|f(x_\delta)|$, and the absolute error, $|x_\delta - \sigma|$ along with the computational order of convergence donated by COC, which can be calculated using the following formula [17]:

$$COC \approx \frac{\ln|\frac{x_{\delta+1}-\sigma}{x_\delta-\sigma}|}{\ln|\frac{x_\delta-\sigma}{x_{\delta-1}-\sigma}|}.$$

Furthermore, the criteria for stopping are as follows:

1. $|x_\delta - \sigma| \leq 10^{-300}$,
2. $|f(x_\delta)| \leq 10^{-300}$.

The basin of attraction is a visual representation technique of how an algorithm behaves based on different starting points [24]. Besides the Kung-Traub conjecture and the efficiency index performance measures, the basin of attraction technique is assumed to be another criterion for performance evaluation [23]. In an ideal situation, the plane is divided into δ basins if a function has δ unique zeroes. For instance, considering the following polynomial $z^2 - 1$ as an example, the roots are $z = 1$ and $z = -1$ [23]. The methods are compared by using a square of $\mathbb{R} \times \mathbb{R} = [-2, 2] \times [-2, 2]$, based on 400×400 points, with a tolerance of $Tol = 0.001$ and a maximum of $N = 20$ iterations. Each point is colored according to the root it converged to, if the recursive process did not converge in a given number of iterations N, then the point will be assigned a black hue. The approach is preferable if there are fewer black spots as the black hue indicates that there has been no convergence on any of the roots after 20 repetitions. The following part presents seven polynomials along with their roots presented in Table 3 to demonstrate the effectiveness of the new proposed iterative schemes namely (KSM) compared with the other three optimal methods which were resolved in the complex domain using Maple 2022.

Table 2. Comparison of several optimal fourth-order recursive schemes with (KSM) methods

Method	Iteration	$\|f(x_\delta)\|$	$\|x_\delta - \sigma\|$	COC
$f(x)_1 = \sin(x) - \frac{x}{2}$				
WM (1.2)	9	4.6804e-423	5.71461e-423	2
KM (3.1)	div	div	div	div
HKM (3.2)	4	3.31913e-308	4.05255e-308	4
LM (3.3)	div	div	div	div
KSM1 (2.15)	5	5.53098e-626	6.75314e-626	4
KSM2 (2.16)	5	2.75967e-626,	3.36947e-626	4
KSM3 (2.17)	5	1.43604e-669	1.75336e-669	4

Method	Iteration	$\lvert f(x_\delta)\rvert$	$\lvert x_\delta - \sigma\rvert$	COC
$f(x)_2 = e^{-x^2+x+2} - 1$				
WM (1.2)	10	5.06655e-539	1.68885e-539	2
KM (3.1)	div	div	div	div
HKM (3.2)	6	1.13413e-628	3.78042e-629	4
LM (3.3)	div	div	div	div
KSM1 (2.15)	5	9.74846e-579	3.24949e-579	4
KSM2 (2.16)	5	1.1225e-540	3.74168e-541	4
KSM3 (2.17)	6	3.3864e-525	1.1288e-525	4
$f(x)_3 = e^{-x^2+x+2} - \cos(x+1) + x^3 + 1$				
WM (1.2)	13	6.31335e-531	1.05223e-531	2
KM (3.1)	31	6.34122e-882	1.05687e-882	4
HKM (3.2)	8	8.67943e-355	1.44657e-355	4
LM (3.3)	34	5.73088e-934	9.55146e-935	4
KSM1 (2.15)	8	3.18149e-898	5.30248e-899	4
KSM2 (2.16)	7	5.85753e-401	9.76255e-402	4
KSM3 (2.17)	9	2.18866e-906	3.64777e-907	4
$f(x)_4 = (x-1)^4 - 1$				
WM (1.2)	11	6.64867e-517	1.66217e-517	2
KM (3.1)	div	div	div	div
HKM (3.2)	7	8.13383e-949	2.03346e-949	4
LM (3.3)	25	1.46473e-317	3.66182e-318	4
KSM1 (2.15)	6	4.00589e-538	1.00147e-538	4
KSM2 (2.16)	6	9.12606e-536	2.28151e-536	4
KSM3 (2.17)	6	2.41513e-423	6.03782e-424	4
$f(x)_5 = \sin(2\cos(x)) - 1 - (x^2) + e^{\sin(x^3)}$				
WM (1.2)	10	3.09351e-566	2.76501e-567	2
KM (3.1)	9	7.41673e-855	6.62915e-856	4
HKM (3.2)	5	4.53229e-419	4.05101e-420	4
LM (3.3)	6	4.65524e-836	4.16090e-837	4
KSM1 (2.15)	5	5.45918e-379	4.87947e-380	4
KSM2 (2.16)	5	3.04174e-373	2.71874e-374	4
KSM3 (2.17)	5	8.11148e-583	7.25013e-584	4

Table 3. The polynomials and their roots in complex domain

Function	Root
$f_1(z) = (z^2-1)^2$	$[\pm 1]$
$f_2(z) = z^3 - z$	$[\pm 1, 0]$
$f_3(z) = (z^3 - z)^2$	$[\pm 1, 0]$
$f_4(z) = z^4 - 1$	$[\pm 1, \pm i]$
$f_5(z) = z^4 - 10z^2 + 9$	$[\pm 3, \pm 1]$
$f_6(z) = z^5 - z$	$[\pm 1, 0, \pm i]$
$f_7(z) = z^6 - 1$	$[\pm 1, \pm 0.5, \pm 0.866025i]$

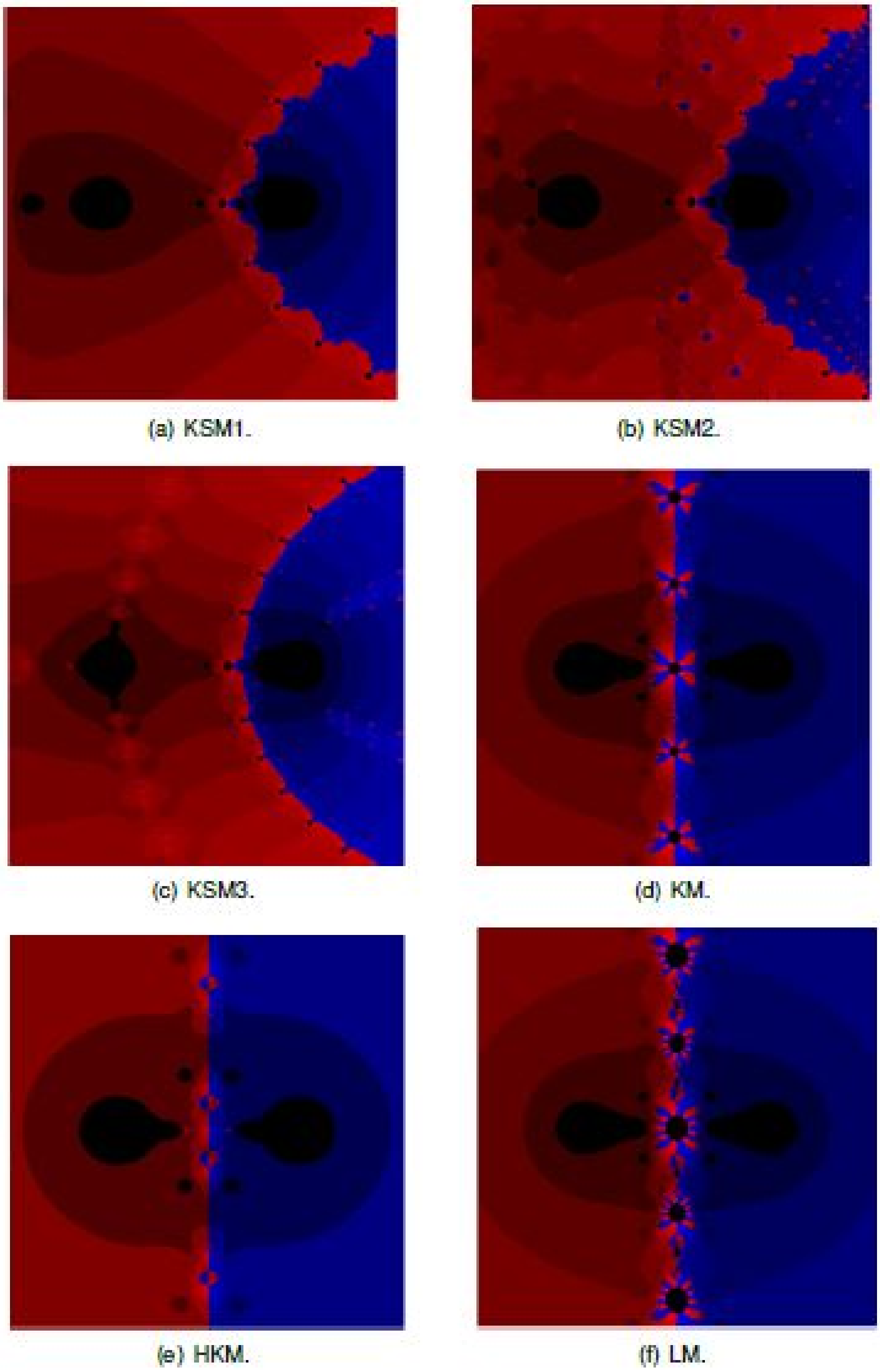


Fig. 1. $f_1(z) = (z^2 - 1)^2$

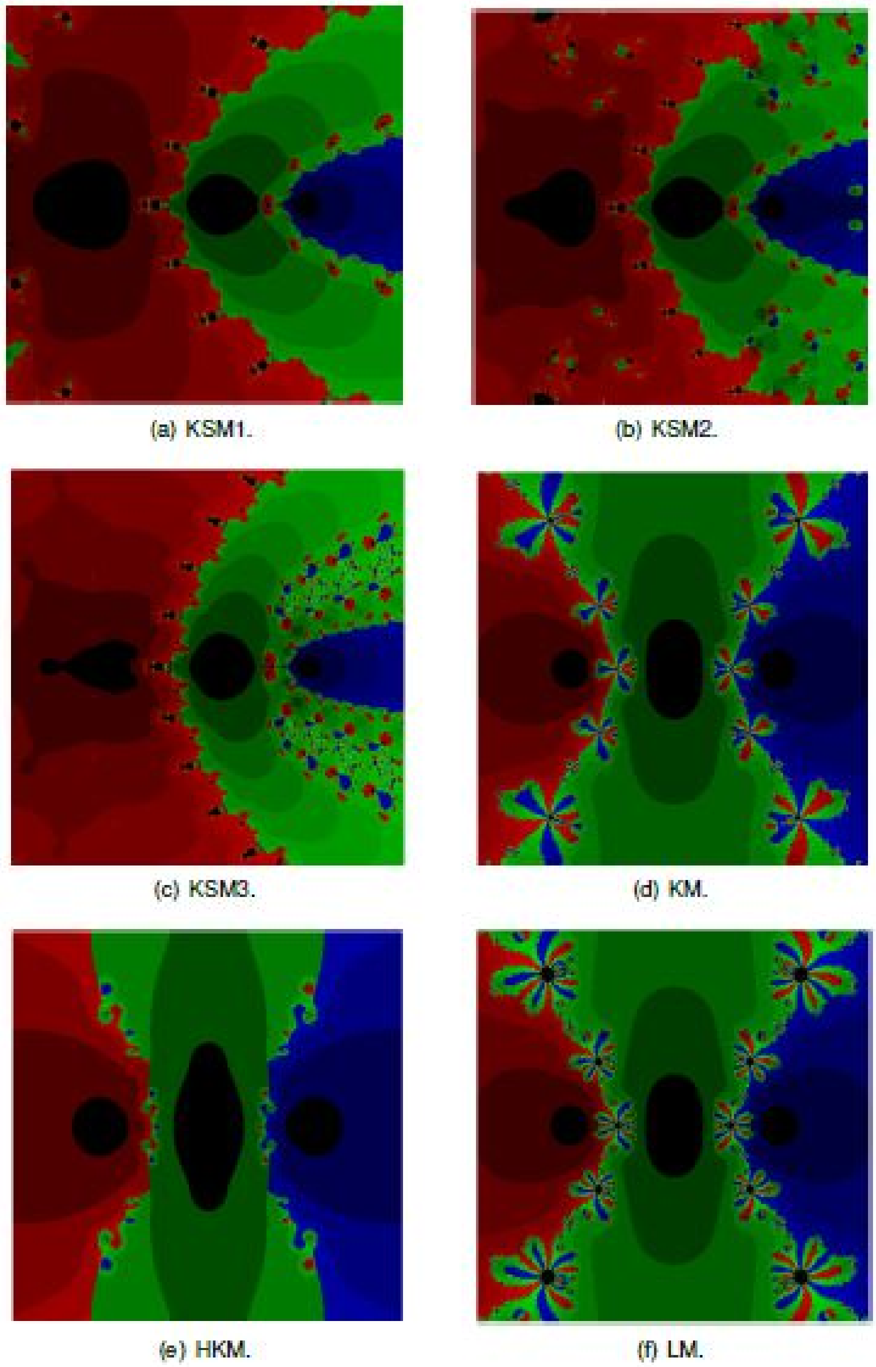
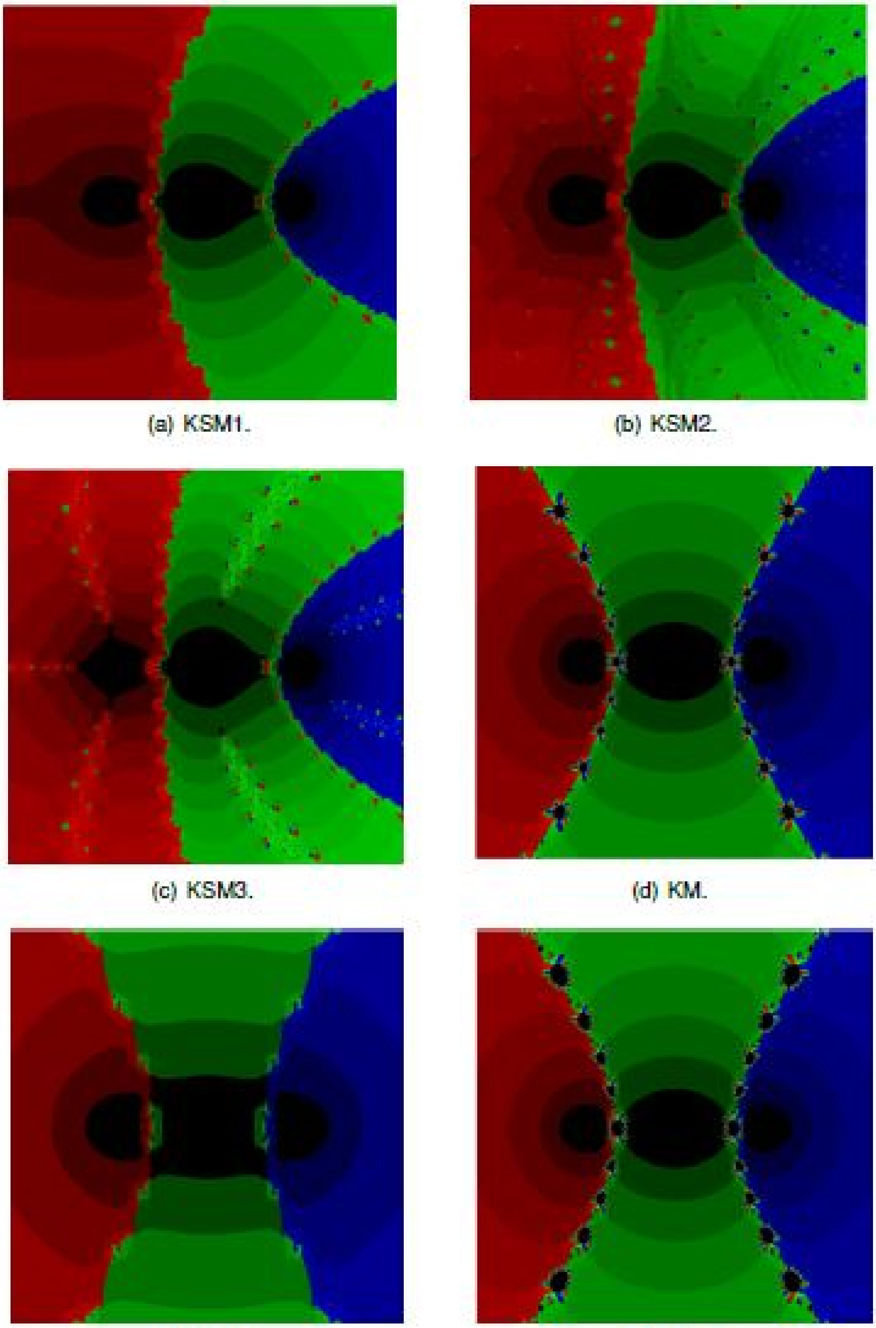


Fig. 2. $f_2(z) = z^3 - z$



(a) KSM1.

(b) KSM2.

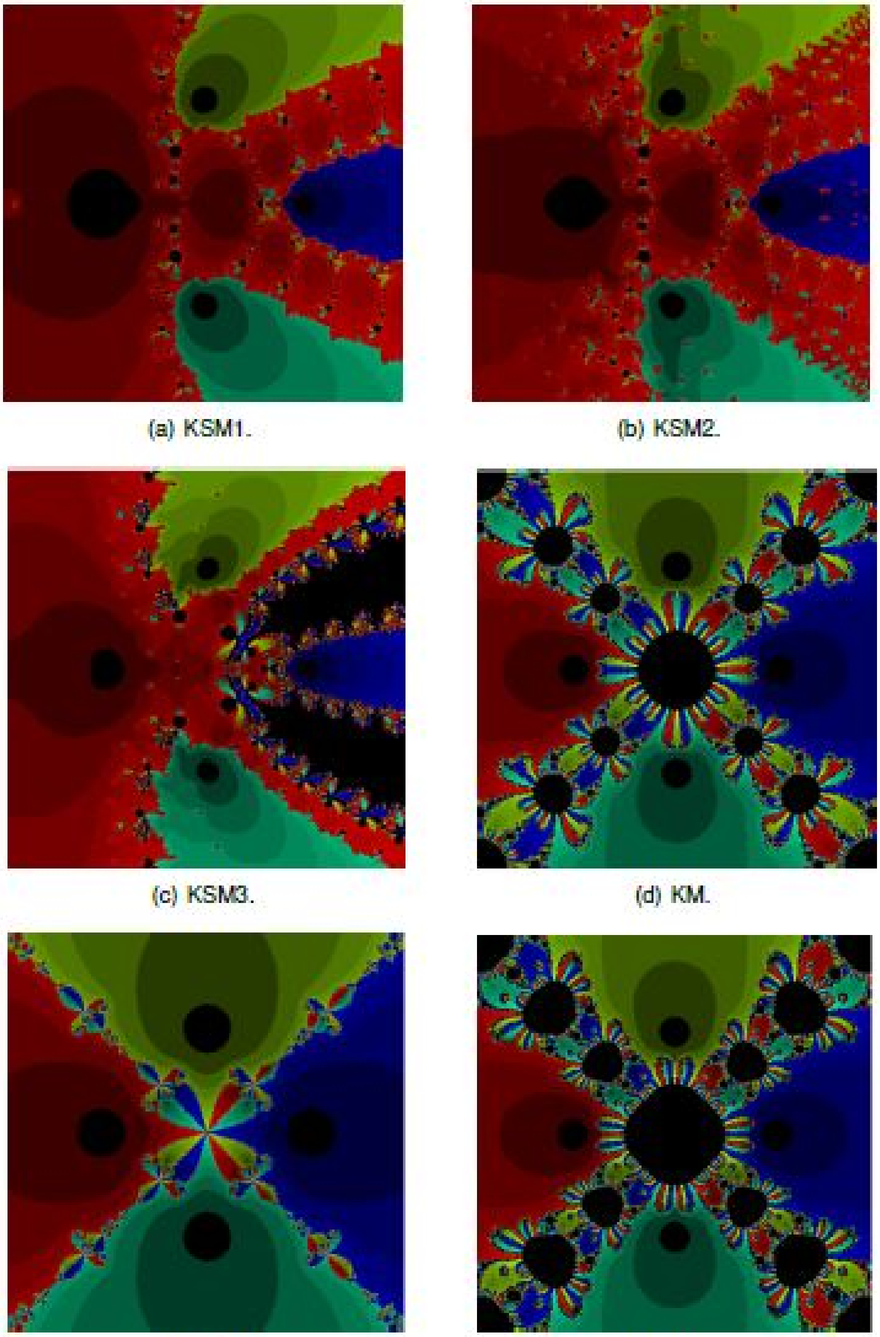
(c) KSM3.

(d) KM.

(e) HKM.

(f) LM.

Fig. 3. $f_3(z) = (z^3 - z)^2$

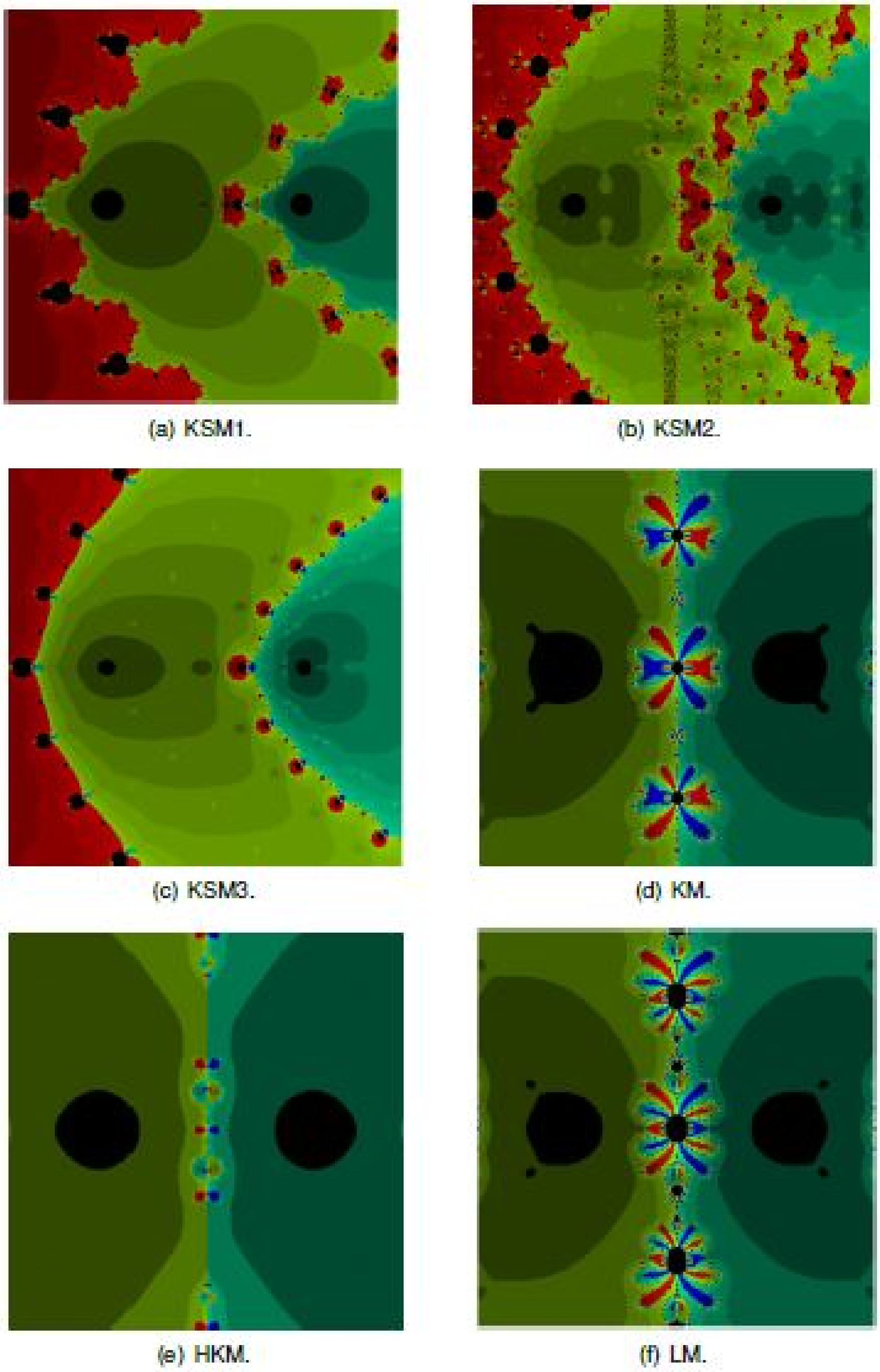


(a) KSM1. (b) KSM2.

(c) KSM3. (d) KM.

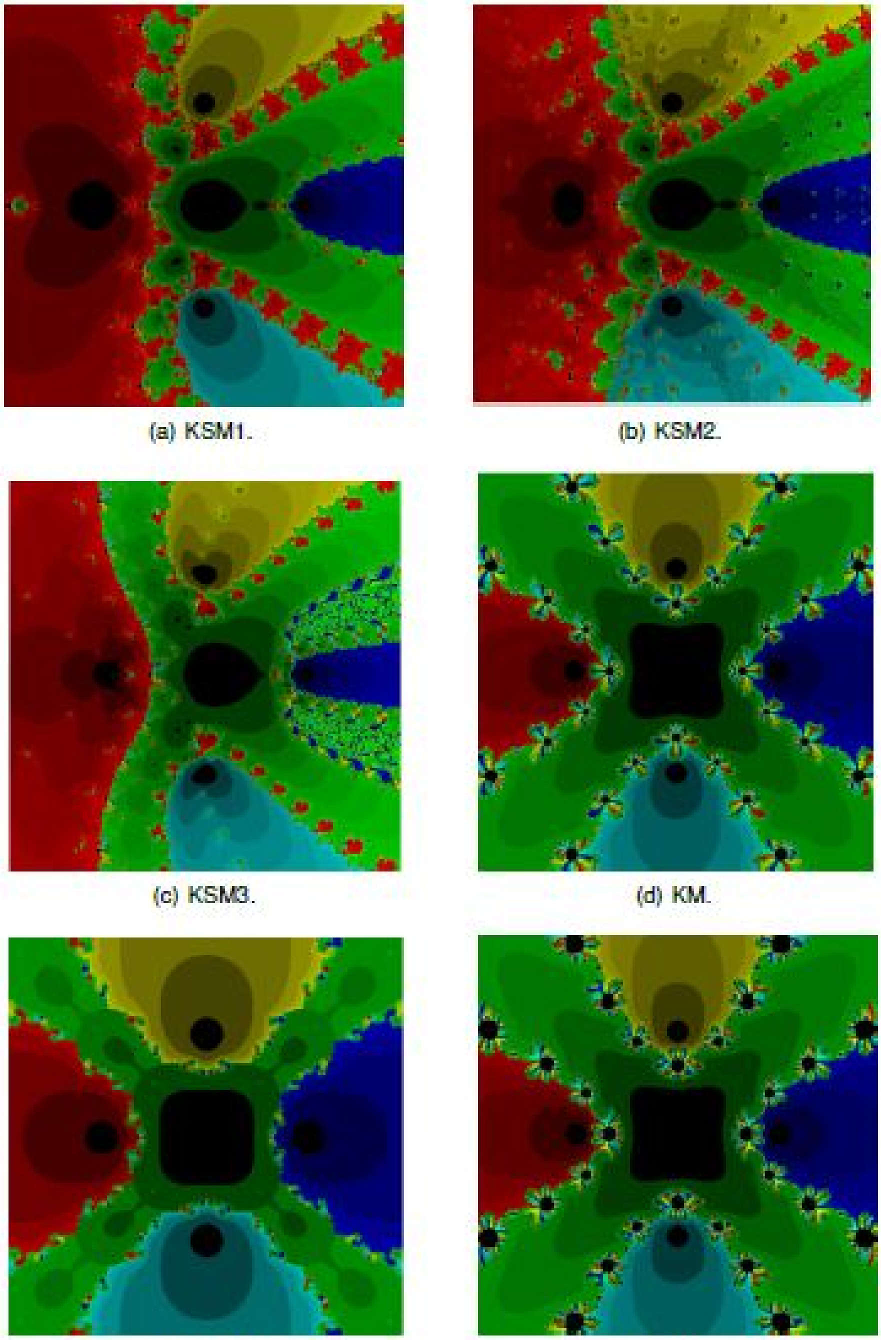
(e) HKM. (f) LM.

Fig. 4. $f_4(z) = z^4 - 1$



(a) KSM1. (b) KSM2. (c) KSM3. (d) KM. (e) HKM. (f) LM.

Fig. 5. $f_5(z) = z^4 - 10z^2 + 9$

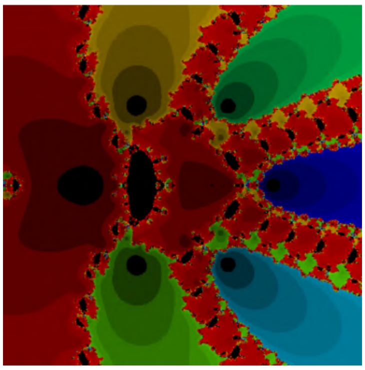


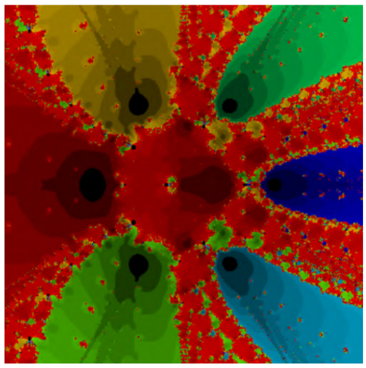
(a) KSM1. (b) KSM2.

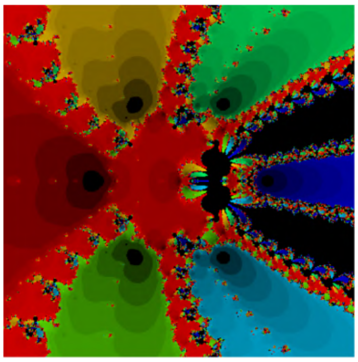
(c) KSM3. (d) KM.

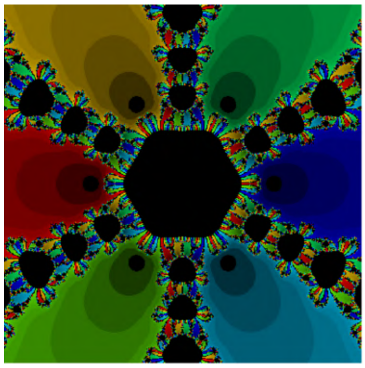
(e) HKM. (f) LM.

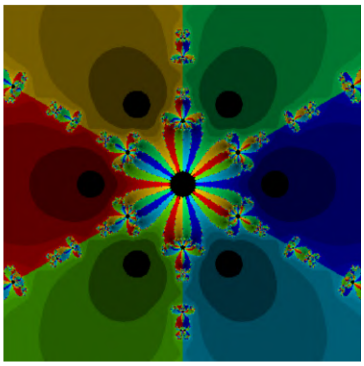
Fig. 6. $f_6(z) = z^5 - z$

[KSM1.] 

[KSM2.] 

[KSM3.] 

[KM.] 

[HKM.] 

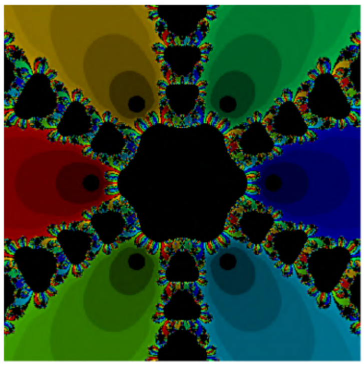
[LM.] 

Fig. 7. $f_7(z) = z^6 - 1$.

4 Conclusion

This work proposes an optimal class of fourth-order algorithms to find a simple zero for nonlinear equations even if the first derivative disappears during the iteration procedure. This theorem was mainly aimed at achieving optimality following the Kung-Traub conjecture, which was successfully sought with the use of Chun approximation to the first derivative of $f(y_\delta)$ and a weight function to increase the order of convergence. Ultimately, the success of the proposed algorithm was demonstrated by evaluating the iterative approaches of equal order of convergence in numerous numerical examples listed in Table 2. In most cases, the new optimal class (KSM) tended to provide better or similar results than the provided methods. Also by using the basin of attraction as another criterion for performance evaluation the newly proposed method showed fewer black hues compared to the other schemes, which is a sign of sufficient performance.

Competing Interests

Authors have declared that no competing interests exist.

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