



Some New Solutions of the Landau-Lifshitz Equation in Cylindrical Symmetric System

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

This paper is devoted to obtaining various explicit dynamic solutions of the Landau-Lifshitz equation in 2-dimensional cylindrical symmetric system. By suitably establishing explicit transformation, we here construct explicit dynamic solutions depending only on the angle θ and time t for Landau-Lifshitz equation with anisotropy field or external magnetic field. Besides, we introduce certain indeterminate solution form of the Landau-Lifshitz equation by solving the indeterminate coefficients to derive explicit magnetic vortex or traveling wave solution. In this paper, we provide some specific examples and attain their some explicit dynamic solutions.

Keywords: Landau-Lifshitz equation; explicit dynamic solution; anisotropy field; external magnetic field; cylindrical symmetric system.

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1 Introduction

The well-known Landau-Lifshitz equation [1] can be written as the form

$$\begin{cases} \frac{\partial \vec{m}}{\partial t} = \lambda_1 \vec{m} \times \vec{H}^e - \lambda_2 \vec{m} \times (\vec{m} \times \vec{H}^e), & \Omega \times (0, T), \\ \vec{H}^e := -\vec{H}(\vec{m}) + \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial \vec{m}}{\partial x_i}) + \vec{H}, & \Omega \times (0, T), \end{cases} \quad (1.1)$$

in which $\vec{m}(x, t) = (m_1, m_2, m_3) : \Omega \times (0, T) \rightarrow S^2 \subset R^3$ is a three-dimensional vector valued unknown function with respect to space variables $x = (x_1, x_2, \dots, x_n)$ and time t ; λ_1 and $\lambda_2 (> 0)$ are constants in physics; the effective magnetic field \vec{H}^e is a conjugate of the anisotropy field $\vec{H}(\vec{m})$, the external magnetic field \vec{H} and the exchange field $\Delta \vec{m} = \sum_{i,j=1}^3 \frac{\partial}{\partial x_j} (a_{ij} \frac{\partial \vec{m}}{\partial x_i})$. Originally introduced by Landau and Lifshitz in 1935, the above motion equation describes the evolution of spin chains in continuum ferromagnets. The Eq. (1.1) plays an essential role in the studying of nonequilibrium magnetism.

Many properties of two-dimensional and especially n-dimensional Landau-Lifshitz equation have been much less studied. The complete form of cylindrical symmetric case for the Heisenberg system [2] can be written as:

$$\vec{m}_t = \vec{m} \times \vec{m}_{rr} + \frac{n-1}{r} \vec{m} \times \vec{m}_r + \frac{n-1}{r^2} \vec{m} \times \vec{m}_{\theta\theta} + \vec{m} \times \vec{m}_{zz}, n \geq 2, \quad (1.2)$$

where $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. In 1990, M. Lakshmanan and K. Porsezian [3] initially proposed the cylindrical symmetric form of the multidimensional Landau-Lifshitz equations:

$$\vec{m}_t = \vec{m} \times \vec{m}_{rr} + \frac{n-1}{r} \vec{m} \times \vec{m}_r, n \geq 1, \quad (1.3)$$

where $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. Under the suitable transformation, the Eq. (1.3) is geometrically equivalent to nonlinear Schrödinger map equation [4]. In 1999, Chang, Shatah and Uhlenbeck [5] considered the 2-dimensional cylindrical symmetric Landau-Lifshitz equations:

$$\vec{m}_t = \vec{m} \times \vec{m}_{rr} + \frac{1}{r} \vec{m} \times \vec{m}_r, n = 2, \quad (1.4)$$

with initial value problem, where $r = \sqrt{x_1^2 + x_2^2}$. When the Eq. (1.4) satisfies the small energy initial condition, they have proved that there exists one global smooth solution to the Eq. (1.4). However, it is extremely important to investigate this open problem that there exists global smooth solution to the n-dimensional Landau-Lifshitz equation. In 2000-2001, B. Guo and G. Yang et al. [6] constructed some exact blow-up solutions to n-dimensional cylindrical symmetric Landau-Lifshitz equations:

$$\vec{m}_t = \vec{m} \times \vec{m}_{rr} + \frac{n-1}{r} \vec{m} \times \vec{m}_r, n \geq 2, \quad (1.5)$$

where $r = \sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$. In 2001, it was initially constructed by B. Guo and G. Yang [7] to obtain some exact nontrivial global smooth solutions on unit sphere for vanishing external magnetic field Eq. (1.4). Although many explicit dynamic solutions (depend only on the radius r and time t) of the Landau-Lifshitz equation have been intensively constructed by many investigators, we are still of the greatest interest in some dynamic solutions (depend only on the angle θ and time t) to the Landau-Lifshitz equation. Meanwhile, some time periodic solutions (depend on the radius r , the angle θ and time t) are known as magnetic vortex solutions (see [8],[9],[10]), which bear a vital role in the understanding of the Landau-Lifshitz flow.

Our main purpose in this paper is to construct various explicit dynamic solutions of the Landau-Lifshitz equation. We mainly consider these explicit solutions in two-dimensional cylindrical symmetric

case. It is important to note that how to choose the suitable first orthogonal matrix is a key step to achieve explicit transform.

In Section 2, our purpose is to obtain explicit dynamic solutions with respect to angle θ and time t for the HLL with a uniaxial anisotropy field $\vec{H}(\vec{m}) = (0, 0, f(m_3(t), \lambda, t))$, in which m_3 depends only on time t . In order to easily construct these solutions, we provide an explicit transformation between the HLX and the HLL.

In Section 3, we provide a meaningful magnetic vortex solution to the Eq. (3.1) with anisotropy field and consider the energy gradient modulus to gain Landau-Lifshitz energy. Furthermore, we also construct a traveling wave solution (depends only on the angle θ and time t) to the Eq. (3.2) with external magnetic field.

In Section 4, we will construct some explicit dynamic smooth solutions to two-dimensional HLZ in cylindrical coordinates, when $\alpha(t)$ is continuous.

2 The Explicit Dynamic Solutions of the HLL

In this section, taking $\lambda_1 = 1$ and $\lambda_2 = 0$ for the Eq.(1.1), we consider the initial-boundary value problem of the Landau-Lifshitz equation under non-vanishing uniaxial anisotropy field

$$HLL : \begin{cases} \frac{\partial \vec{m}}{\partial t} = \vec{m} \times (\Delta \vec{m} + f(m_3(t), \lambda, t) \vec{e}_3), & (x, t) \in \Omega \times (0, T), \\ \vec{m}(x, 0) = \vec{\varphi}(x), & x \in \Omega, \\ \vec{m}(x, t) = \vec{\phi}(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ \vec{m} \in S^2 \subset R^3, & \text{in } R^n \times [0, \infty), \end{cases}$$

where $\vec{e}_3 = (0, 0, 1)$. When $f(m_3(t), \lambda, t)$ takes 0, the Landau-Lifshitz equation with isotropic case will be written as follows:

$$HLX : \begin{cases} \frac{\partial \vec{n}}{\partial t} = \vec{n} \times \Delta \vec{n}, & (x, t) \in \Omega \times (0, T), \\ \vec{n}(x, 0) = \vec{\varpi}(x), & x \in \Omega, \\ \vec{n}(x, t) = \vec{\omega}(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ \vec{n} \in S^2 \subset R^3, & \text{in } R^n \times [0, \infty). \end{cases}$$

In the present section, we will construct explicit dynamic solutions to the HLL in 2-dimensional cylindrical coordinates. Constructing explicit dynamic solution to n-dimensional HLL is still an open and important problem. In order to obtain these explicit solutions of the HLL, the following Lemmas are firstly established.

Lemma 2.1. *Let $g = f(m_3(t), \lambda, t)$ satisfy the 3×3 matrix*

$$H = \begin{pmatrix} 0 & -g & 0 \\ g & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then the first orthogonal matrix $A(t)$ defined by

$$A(t) = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that is an explicit solution of the equation

$$\frac{\partial A(t)}{\partial t} = -HA(t),$$

in which $\varphi = \int_0^t f(m_3(s), \lambda, s) ds$.

Proof. We can easily verify that $A(t)$ satisfies

$$\frac{\partial A(t)}{\partial t} = \begin{pmatrix} -g \sin \varphi & g \cos \varphi & 0 \\ -g \cos \varphi & -g \sin \varphi & 0 \\ 0 & 0 & 0 \end{pmatrix} = -HA(t).$$

□

Next the solution of the HLL considered, we gain some explicit dynamic solutions that include in $C^1([0, \infty); C^2(R^n; R^3))$ and are classical. By using above lemma 2.1, we construct a transformation from static solution of the HLX to explicit dynamic solution of the HLL, as follows:

Lemma 2.2. *Let $g = f(m_3(t), \lambda, t) \in C(0, T)(0 < T \leq \infty)$, $\vec{\varphi} \in C^2(\Omega)$ and $\vec{\phi} \in C^1(0, T; C^2(\partial\Omega; R^3))$, then $\vec{m} \in C^1(0, T; C^2(R^n; R^3))$ is a solution of the HLL if and only if*

$$\vec{n} = \vec{m}A(t)$$

is a solution of the HLX, where $\vec{\omega} = \vec{\varphi}A(0) = \vec{\varphi}A_0$ and $\vec{\omega} = \vec{\phi}A(t)$.

Proof. i) For one thing, assuming that \vec{m} is a solution of the HLL, we have

$$\begin{aligned} & \frac{\partial \vec{n}}{\partial t} - \vec{n} \times \Delta \vec{n} \\ &= \frac{\partial(\vec{m}A(t))}{\partial t} - (\vec{m}A(t) \times \Delta(\vec{m}A(t))) \\ &= \vec{m} \frac{\partial(A(t))}{\partial t} + (\vec{m} \times (0, 0, g))A(t) \\ &= \vec{m} \frac{\partial(A(t))}{\partial t} + \vec{m}HA(t) \\ &= 0. \end{aligned}$$

Thus, this proves $\vec{n} = \vec{m}A(t)$ is a solution of the HLX;

ii) For another thing, supposing that $\vec{n} = \vec{m}A(t)$ is a solution of the HLX, we deduce that

$$\begin{aligned} & \frac{\partial \vec{m}}{\partial t} - \vec{m} \times (\Delta \vec{m} + (0, 0, g)) \\ &= \frac{\partial(\vec{n}(A(t))^T)}{\partial t} - (\vec{n}(A(t))^T \times (\Delta(\vec{n}(A(t))^T) + (0, 0, g))) \\ &= \vec{n} \frac{\partial((A(t))^T)}{\partial t} - (\vec{n}(A(t))^T \times (0, 0, g)) \\ &= \vec{n} \frac{\partial((A(t))^T)}{\partial t} + (\vec{n}(A(t))^T H^T) \\ &= 0. \end{aligned}$$

Therefore, this proves that $\vec{m} = \vec{n}(A(t))^T$ is a solution of the HLL.

□

Remark 2.3. *In Lemma 2.2, we construct the transform that is explicit form. When the solution to the HLX is given by us, of course, we will easily calculate the solution to the HLL. In view of explicit dynamic solution (depends only on the angle θ and time t) for the HLL, the first orthogonal matrix $A(t)$ ensures to achieve an explicit transform between static solution of the HLX and explicit dynamic solution of the HLL. As a matter of fact, we introduce $\varphi = \int_0^t f(m_3(s), \lambda, s)ds$ that is continuous on $(0, T)$ and $\int_0^t f(m_3(s), \lambda, s)ds = \int_0^t f(n_3(s), \lambda, s)ds$, in which n_3 is independent of x , thus $\vec{m} = \vec{n}(A(t))^T$ can be easily obtained.*

According to Lemma 2.2, some explicit dynamic solutions with respect to radius r and time t have been abundantly constructed by Guo and Yang [11]. Nevertheless, it is really significant to construct some explicit dynamic solutions (depend only on the angle θ and time t) of the HLL.

Theorem 2.4. Let $\theta = \arctan(\frac{x_2}{x_1})$, we have

$$\vec{m}(\theta, t) = \begin{pmatrix} \cos(\theta - \varphi) \\ \sin(\theta - \varphi) \\ 0 \end{pmatrix}^T \tag{2.1}$$

that is an explicit dynamic solution for 2-dimensional HLL in cylindrical coordinates, and satisfies the initial condition

$$\vec{m}(\theta, 0) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}^T,$$

where $\varphi = \int_0^t f(m_3(s), \lambda, s)ds$ and $\vec{m}(\theta, t) \in S^2 \cap C^\infty([0, \infty) \times [0, 2\pi])$.

Proof. We have constructed static cylindrical symmetric solution of the HLX in the following form

$$\vec{n}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}^T.$$

According to Lemma 2.2, we obtain

$$\vec{m}(\theta, t) = \vec{n}(A(t))^T = \begin{pmatrix} \cos(\theta - \varphi) \\ \sin(\theta - \varphi) \\ 0 \end{pmatrix}^T$$

that is an explicit dynamic solution for the HLL with the initial value above. □

Remark 2.5. Let $A_1 \in R^{3 \times 3}$ be the first constant orthogonal matrix i.e. $|A_1| = 1$, in which

$$A_1 = \begin{pmatrix} \pm\sqrt{1-C_1^2} & C_1 & 0 \\ C_1 & \mp\sqrt{1-C_1^2} & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

satisfying $|C_1| \leq 1$. If \vec{n} is a static solution for the HLX, then $\vec{n}A_1B^T(t)$ is a dynamic solution for the HLL, where

$$B(t) = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\phi = \int_0^t f(m_3(s), \lambda, s)ds$.

Remark 2.6. Let $A_2 \in R^{3 \times 3}$ be the first constant orthogonal matrix i.e. $|A_2| = 1$, in which

$$A_2 = \begin{pmatrix} \pm\sqrt{1-C_2^2} & C_2 & 0 \\ -C_2 & \pm\sqrt{1-C_2^2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

satisfying $|C_2| \leq 1$. If \vec{m} is an explicit dynamic solution to the HLL, then $\vec{m}A_2$ is also an explicit dynamic solution to the HLL.

Example 2.7. On the case of 2-dimensional Landau-Lifshitz equation with a uniaxial anisotropy field $(0, 0, 2\lambda(1 - m_3)^3)$, explicit static solution has been derived by Papanicolaou and Zakrezewski [12]. This equation may be recast as follows:

$$\begin{cases} \frac{\partial \vec{m}}{\partial t} = \vec{m} \times (\Delta \vec{m} + (0, 0, 2\lambda(1 - m_3)^3)), & \text{in } R^n \times (0, \infty), \\ \vec{m} \in S^2, & \text{in } R^n \times [0, \infty). \end{cases} \tag{2.2}$$

It follows from Theorem 2.4 that

$$\begin{aligned}\varphi &= \int_0^t f(n_3(s), \lambda, s) ds \\ &= \int_0^t f(m_3(s), \lambda, s) ds \\ &= \int_0^t 2\lambda(1-0)^3 ds \\ &= 2\lambda t;\end{aligned}$$

therefore, we gain

$$\vec{m}(\theta, t) = \begin{pmatrix} \cos(\theta - 2\lambda t) \\ \sin(\theta - 2\lambda t) \\ 0 \end{pmatrix}^T \quad (2.3)$$

that is an explicit dynamic solution of 2-dimensional Eq.(2.2) in cylindrical symmetric system, and satisfies the initial value

$$\vec{m}(\theta, 0) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}^T.$$

3 The Vortex or Traveling Wave Solution of the Eq. (3.1) and Eq. (3.2)

In this section, we construct vortex solution of the Landau-Lifshitz equation with uniaxial anisotropy field

$$\vec{u}_t = \vec{u} \times \vec{u}_{rr} + \frac{1}{r} \vec{u} \times \vec{u}_r + \frac{1}{r^2} \vec{u} \times \vec{u}_{\theta\theta} + \vec{u} \times \vec{H}(\vec{u}), \quad (3.1)$$

and traveling wave solution (see [13],[14]) of the Landau-Lifshitz equation with external magnetic field

$$\vec{v}_t = \vec{v} \times \vec{v}_{rr} + \frac{1}{r} \vec{v} \times \vec{v}_r + \frac{1}{r^2} \vec{v} \times \vec{v}_{\theta\theta} + \vec{v} \times \alpha(t) \vec{H}, \quad (3.2)$$

where $\vec{H}(\vec{u}) = (0, 0, \lambda u_3)$, $\vec{H} = (0, 0, h_3)$ and $\alpha(t)$ satisfies $\int_0^t \alpha(s) ds \in C[0, T]$, $0 < T \leq +\infty$.

Theorem 3.1. Let $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(\frac{x_2}{x_1})$ and $\varphi(r) = 2\arctan\sqrt{\frac{-b^2+p^2+\lambda r^2+\omega r^2}{-b^2+p^2+\lambda r^2-\omega r^2}}$. Then we have

$$\vec{u}(r, \theta, t) = \begin{pmatrix} \sin(\varphi(r))\cos(p\theta + \omega t + \xi) \\ \sin(\varphi(r))\sin(p\theta + \omega t + \xi) \\ \cos(\varphi(r)) \end{pmatrix}^T \quad (3.3)$$

that is a magnetic vortex solution of 2-dimensional Eq. (3.1) in cylindrical symmetric system.

Proof. If $\vec{u}(r, \theta, t) = (u_1(r, \theta, t), u_2(r, \theta, t), u_3(r, \theta, t))$ is a magnetic vortex solution for the Eq.(3.1), where

$$\begin{cases} u_1(r, \theta, t) = \sin(\varphi(r))\cos(p\theta + \omega t + \xi), \\ u_2(r, \theta, t) = \sin(\varphi(r))\sin(p\theta + \omega t + \xi), \\ u_3(r, \theta, t) = \cos(\varphi(r)), \end{cases} \quad (3.4)$$

p denotes the vortex degree; ω denotes the frequency; ξ denotes the initial phase, then we deduce from the Eq. (3.1) and (3.4) that

$$-r^2 \frac{d^2}{dr^2}(\varphi(r)) - r \frac{d}{dr}(\varphi(r)) + \left(\frac{p^2}{2} + \frac{\lambda r^2}{2}\right) \sin(2\varphi(r)) + r^2 \omega \sin(\varphi(r)) = 0. \quad (3.5)$$

Next let us introduce function

$$\varphi(r) = 2 \arctan\left(\frac{r^b}{\mu}\right), (\mu \neq 0), \quad (3.6)$$

and substitute (3.6) into the equation (3.5), then we gain

$$b^2\mu^2r^b - b^2r^{3b} - p^2\mu^2r^b + p^2r^{3b} - \lambda\mu^2r^{2+b} + \lambda r^{2+3b} - \omega\mu^2r^{2+b} - \omega r^{2+3b} = 0. \quad (3.7)$$

After simplifying (3.7), we have

$$r^{2b} = \frac{\mu^2(-b^2 + p^2 + \lambda r^2 + \omega r^2)}{-b^2 + p^2 + \lambda r^2 - \omega r^2}.$$

Thus, we obtain

$$\varphi(r) = 2\arctan\sqrt{\frac{-b^2 + p^2 + \lambda r^2 + \omega r^2}{-b^2 + p^2 + \lambda r^2 - \omega r^2}}. \quad (3.8)$$

□

Remark 3.2. We deduce from (3.3) and (3.8) that energy gradient modulus

$$|\vec{u}_r(r, t)|^2 = \frac{4(\omega b^2 - \omega p^2)^2 r^2}{(b^2 - p^2 - \lambda r^2)^2 (b^2 - p^2 - \lambda r^2 + \omega r^2)(b^2 - p^2 - \lambda r^2 - \omega r^2)} \quad (3.9)$$

is independent of t . And $\int_0^R |\vec{u}_r(r, t)|^2 r dr$ (R is constant or infinity) is bounded or unbounded.

Example 3.3. In view of (3.9), we take $R = +\infty$, $\omega = 1$, $p = 2$, $b = 1$, $\lambda = 3$, and obtain

$$|\vec{u}_r(r, t)|^2 = \frac{36r^2}{(-3 - 3r^2)^2 (-3 - 2r^2)(-3 - 4r^2)}. \quad (3.10)$$

Integrating (3.10) over $[0, +\infty)$, we have the energy

$$E = \int_0^{+\infty} |\vec{u}_r(r, t)|^2 r dr = 2 + 2 \ln(3) - 6 \ln(2).$$

If we also take $R = 2$, $\omega = 1$, $p = 1$, $b = 3$, $\lambda = 1$, then we gain

$$|\vec{u}_r(r, t)|^2 = \frac{16r^2}{(8 - r^2)^2 (4 - r^2)}. \quad (3.11)$$

Therefore, we also have the energy

$$E = \int_0^2 \frac{16r^2}{(8 - r^2)^2 (4 - r^2)} r dr = +\infty.$$

Remark 3.4. As is mentioned above, it is the fact that there exists finite energy magnetic vortex solution of the Eq.(3.1) under prescribed boundary data. In addition, if the Landau-Lifshitz equation loses anisotropy field, then magnetic vortex solution will transform traveling wave solution.

Theorem 3.5. Let $\theta = \arctan(\frac{x_2}{x_1})$. There is a traveling wave solution $\vec{v}(\theta, t) = (v_1(\theta, t), v_2(\theta, t), v_3(\theta, t))$ for the Eq.(3.2) with external magnetic field, in which

$$\vec{v}(\theta, t) = \begin{pmatrix} \sin(2 \arctan(1)) \cos(p\theta - \int h_3 \alpha(t) dt + \xi_1) \\ \sin(2 \arctan(1)) \sin(p\theta - \int h_3 \alpha(t) dt + \xi_1) \\ \cos(2 \arctan(1)) \end{pmatrix}^T. \quad (3.12)$$

Proof. We will suppose

$$\begin{cases} v_1(r, \theta, t) = \sin(\varphi(r)) \cos(p\theta + \omega(t)t + \xi), \\ v_2(r, \theta, t) = \sin(\varphi(r)) \sin(p\theta + \omega(t)t + \xi), \\ v_3(r, \theta, t) = \cos(\varphi(r)), \end{cases} \quad (3.13)$$

and deduce from the Eq. (3.2) and (3.13) that

$$-r^2\left(\frac{d^2}{dr^2}\varphi(r)\right) - r\left(\frac{d}{dr}\varphi(r)\right) + \frac{p^2}{2}\sin(2\varphi(r)) + r^2(h_3\alpha(t) + \left(\frac{d}{dt}\omega(t)\right)t + \omega(t))\sin\varphi(r) = 0. \quad (3.14)$$

If we take

$$h_3\alpha(t) + \left(\frac{d}{dt}\omega(t)\right)t + \omega(t) = 0, \quad (3.15)$$

then we obtain an exact solution for the equation (3.15), and it may be written as

$$\omega(t) = \frac{C_1 - \int \alpha(t)h_3 dt}{t}, \quad (3.16)$$

in which C_1 is constant. Therefore, (3.13) can be rewritten as

$$\begin{cases} v_1(r, \theta, t) = \sin(\varphi(r))\cos(p\theta - \int h_3\alpha(t)dt + \xi_1), \\ v_2(r, \theta, t) = \sin(\varphi(r))\sin(p\theta - \int h_3\alpha(t)dt + \xi_1), \\ v_3(r, \theta, t) = \cos(\varphi(r)), \end{cases} \quad (3.17)$$

in which ξ_1 is constant. Substituting (3.17) into the Eq. (3.2), we easily obtain

$$-r^2\frac{d^2}{dr^2}(\varphi(r)) - r\frac{d}{dr}(\varphi(r)) + \frac{p^2}{2}\sin(2\varphi(r)) = 0. \quad (3.18)$$

Let us define function

$$\varphi(r) = 2\arctan\left(\frac{ar^\beta}{\mu}\right), (\mu \neq 0), \quad (3.19)$$

where β, a, μ are constant.

Thus, we deduce from (3.18) and (3.19) that

$$\varphi(r) = 2\arctan(1). \quad (3.20)$$

□

4 The Explicit Dynamic Solutions of the HLZ

In this section, we consider the initial-boundary value problem of the Landau-Lifshitz equation under non-vanishing external magnetic field

$$HLZ : \begin{cases} \frac{\partial \vec{m}}{\partial t} = \vec{m} \times (\Delta \vec{m} + \alpha(t)\vec{H}), & (x, t) \in \Omega \times (0, T), \\ \vec{m}(x, 0) = \vec{\phi}(x), & x \in \Omega, \\ \vec{m}(x, t) = \vec{\psi}(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ \vec{m} \in S^2 \subset R^3, & \text{in } R^n \times [0, \infty), \end{cases}$$

and the initial-boundary value problem for the Landau-Lifshitz equation under vanishing external magnetic field

$$HLY : \begin{cases} \frac{\partial \vec{n}}{\partial t} = \vec{n} \times \Delta \vec{n}, & (x, t) \in \Omega \times (0, T), \\ \vec{n}(x, 0) = \vec{\xi}(x), & x \in \Omega, \\ \vec{n}(x, t) = \vec{\zeta}(x, t), & (x, t) \in \partial\Omega \times (0, T), \\ \vec{n} \in S^2 \subset R^3, & \text{in } R^n \times [0, \infty), \end{cases}$$

where $\vec{H} = (h_1, h_2, h_3)$, and h_1, h_2, h_3 are constant, and $\alpha(t)$ is continuous.

Next we construct some explicit dynamic solutions in respect to angle θ and time t to the HLZ. Considering two cases, we let $h_2^2 + h_3^2 \neq 0$ and $h_2^2 + h_3^2 = 0$ in this section. In order to conveniently get explicit dynamic solution of the HLZ, we give some Lemmas as follows:

Lemma 4.1. *Let*

i) $h_2^2 + h_3^2 \neq 0$ and $h_1 = h_2 = 0$, we have $f = \alpha(t) |\vec{H}|$ that satisfies the 3×3 matrix

$$H_1^* = \begin{pmatrix} 0 & -f & 0 \\ f & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix};$$

ii) $h_2^2 + h_3^2 = 0$ and $h_2 = h_3 = 0$, we have the 3×3 matrix in the following form

$$H_2^* = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -g \\ 0 & g & 0 \end{pmatrix},$$

in which $g = \alpha(t) |\vec{H}|$;

then

iii) the first orthogonal matrix $F_1(t)$ defined by

$$F_1(t) = \begin{pmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

that is an explicit solution of the equation

$$\frac{\partial F_1(t)}{\partial t} = -H_1^* F_1(t),$$

in which $|\vec{H}| = h_3$ and $\psi = \left| \vec{H} \right| \int_0^t \alpha(s) ds$;

iv) the first orthogonal matrix $F_2(t)$ defined by

$$F_2(t) = \begin{pmatrix} 0 & 0 & 1 \\ -\sin \phi & \cos \phi & 0 \\ -\cos \phi & -\sin \phi & 0 \end{pmatrix}$$

that is an explicit solution of the equation

$$\frac{\partial F_2(t)}{\partial t} = -H_2^* F_2(t),$$

in which $|\vec{H}| = h_1$ and $\phi = \left| \vec{H} \right| \int_0^t \alpha(s) ds$.

Proof. We can easily verify that $F_1(t)$ is the first orthogonal matrix satisfying

$$\frac{\partial F_1(t)}{\partial t} = \begin{pmatrix} -\alpha(t)h_3 \sin \psi & \alpha(t)h_3 \cos \psi & 0 \\ -\alpha(t)h_3 \cos \psi & -\alpha(t)h_3 \sin \psi & 0 \\ 0 & 0 & 0 \end{pmatrix} = -H_1^* F_1(t).$$

Similarly, we can prove that

$$\frac{\partial F_2(t)}{\partial t} = \begin{pmatrix} 0 & 0 & 0 \\ -\alpha(t)h_1 \cos \phi & -\alpha(t)h_1 \sin \phi & 0 \\ \alpha(t)h_1 \sin \phi & -\alpha(t)h_1 \cos \phi & 0 \end{pmatrix} = -H_2^* F_2(t).$$

□

Lemma 4.2. Let $\alpha(t) \in C[0, T]$, $\vec{\phi} \in C^2(\Omega)$ and $\vec{\psi} \in C^1([0, T]; C^2(\partial\Omega; R^3))$, then $\vec{m} \in C^1([0, T]; C^2(R^n; R^3))$ is a solution of the HLZ if and only if

i)

$$\vec{n} = \vec{m}F_1(t)$$

is a solution of the HLY, where $\vec{\xi} = \vec{\phi}P_1$ and $\vec{\zeta} = \vec{\psi}F_1(t)$;

ii)

$$\vec{n} = \vec{m}F_2(t)$$

is a solution of the HLY, where $\vec{\xi} = \vec{\phi}P_2$ and $\vec{\zeta} = \vec{\psi}F_2(t)$.

Proof. For the proof of i), assume that $\vec{n}(\theta) = \vec{m}(\theta, t)F_1(t)$, then $\alpha(t) \in C[0, T]$ shows that $\psi(t) = h_3 \int_0^t \alpha(s)ds$ is continuously differentiable on $[0, T]$. We have

$$\vec{m} \in C^1([0, T]; C^2(R^n; R^3))$$

if and only if

$$\vec{n} \in C^2(R^n; R^3).$$

Let $\psi(0) = 0$, we know $F_1(0) = P_1$. So for every $\theta \in \Omega$, $\vec{m}(\theta, 0) = \vec{\phi}(\theta)$ if and only if $\vec{n}(\theta) = \vec{\phi}(\theta)P_1$, and for every $(\theta, t) \in \partial\Omega \times (0, T)$, $\vec{m}(\theta, t) = \vec{\psi}(\theta, t)$ if and only if $\vec{n}(\theta) = \vec{\psi}(\theta, t)F_1(t)$. Next performing the similar proof of Lemma 2.2, we complete the proof of i). Meanwhile, the same procedure may be easily adapted to complete the proof of ii). Thus, the proof of Lemma 4.2 is completed. \square

According to above Lemma 4.2, some explicit exact solutions (depend only on the radius r and time t) have been obtained by Guo and Yang [15]. But we are interested in explicit dynamic solution with respect to angle θ and time t for the HLZ. Here we will derive some explicit dynamic solutions to the HLZ by using above Lemma 4.2.

Theorem 4.3. Let $\theta = \arctan(\frac{x_2}{x_1})$ and $\psi(t) = h_3 \int_0^t \alpha(s)ds$. Then $\vec{m}(\theta, t) \in S^2 \cap C^\infty([0, \infty) \times [0, 2\pi])$ defined by

$$\vec{m}(\theta, t) = \begin{pmatrix} \cos(\theta - \psi) \\ \sin(\theta - \psi) \\ 0 \end{pmatrix}^T \tag{4.1}$$

that is an explicit dynamic solution of 2-dimensional HLZ in cylindrical symmetric system, satisfying the following initial-boundary conditions:

i) initial condition:

$$\vec{m}(\theta, 0) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}^T,$$

where $\theta \in \Omega = [0, 2\pi]$;

ii) boundary conditions:

$$\vec{m}(0, t) = \begin{pmatrix} \cos(-\psi) \\ \sin(-\psi) \\ 0 \end{pmatrix}^T,$$

where $(0, t) \in \{0\} \times [0, \infty)$, and

$$\vec{m}(2\pi, t) = \begin{pmatrix} \cos(2\pi - \psi) \\ \sin(2\pi - \psi) \\ 0 \end{pmatrix}^T.$$

where $(2\pi, t) \in \{2\pi\} \times [0, \infty)$.

Proof. Since we constructed static solution to the HLY in the following form

$$\vec{n}(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}^T,$$

using Lemma 4.2, we obtain

$$\vec{m}(\theta, t) = \vec{n}(F_1(t))^T = \begin{pmatrix} \cos(\theta - \psi) \\ \sin(\theta - \psi) \\ 0 \end{pmatrix}^T$$

that is an explicit dynamic solution of the HLZ with above initial value and boundary values. \square

Theorem 4.4. Let $\theta = \arctan(\frac{x_2}{x_1})$ and $\phi(t) = h_1 \int_0^t \alpha(s) ds$. We have $\vec{m}(\theta, t) \in S^2 \cap C^\infty([0, \infty) \times [0, 2\pi])$ defined by

$$\vec{m}(\theta, t) = \begin{pmatrix} 0 \\ \sin(\theta - \phi) \\ -\cos(\theta - \phi) \end{pmatrix}^T \tag{4.2}$$

that is an explicit dynamic solution of 2-dimensional HLZ in cylindrical coordinates, and satisfies the following initial-boundary conditions:

i) initial condition:

$$\vec{m}(\theta, 0) = \begin{pmatrix} 0 \\ \sin \theta \\ -\cos \theta \end{pmatrix}^T,$$

where $\theta \in \Omega = [0, 2\pi]$;

ii) boundary conditions:

$$\vec{m}(0, t) = \begin{pmatrix} 0 \\ \sin(-\phi) \\ -\cos(-\phi) \end{pmatrix}^T,$$

where $(0, t) \in \{0\} \times [0, \infty)$, and

$$\vec{m}(2\pi, t) = \begin{pmatrix} 0 \\ \sin(2\pi - \phi) \\ -\cos(2\pi - \phi) \end{pmatrix}^T,$$

where $(2\pi, t) \in \{2\pi\} \times [0, \infty)$.

Proof. Since we constructed static solution of the HLY in the following expression:

$$\vec{n}(\theta) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}^T,$$

using Lemma 4.2 again, we gain

$$\vec{m}(\theta, t) = \vec{n}(F_2(t))^T = \begin{pmatrix} 0 \\ \sin(\theta - \phi) \\ -\cos(\theta - \phi) \end{pmatrix}^T$$

that is an explicit dynamic solution of the HLZ with above initial value and boundary values. \square

Remark 4.5. Let

$$F = \begin{pmatrix} C & \pm\sqrt{1-C^2} & 0 \\ \mp\sqrt{1-C^2} & C & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

be the third-order first constant orthogonal matrix, and satisfy $|C| \leq 1$, here if \vec{n} is a solution of the HLY, we conclude that $\vec{m} = \vec{n}(F_2(t)F)^T = \vec{n}(F_2(t)F)^{-1}$ is an explicit dynamic solution of the HLZ.

Example 4.6. In Nakamura and Sasada [16], authors studied the HLZ with external magnetic field in 1974. On the system of two-dimensional cylindrical symmetric model, if we take $H = (0, 0, h)$ and $\alpha(t) \equiv 1$, the HLZ can be written as

$$\begin{cases} \frac{\partial \vec{m}}{\partial t} = \vec{m} \times (\Delta \vec{m} + (0, 0, h)), & \text{in } R^n \times (0, \infty), \\ \vec{m} \in S^2, & \text{in } R^n \times [0, \infty), \end{cases} \quad (4.3)$$

in which h is nonzero constant.

According to Theorem 4.3, we construct explicit dynamic solution $\vec{m}(\theta, t) = (m_1(\theta, t), m_2(\theta, t), m_3(\theta, t))$ of the Eq.(4.3), in which

$$\vec{m}(\theta, t) = \begin{pmatrix} \cos(\theta - ht) \\ \sin(\theta - ht) \\ 0 \end{pmatrix}^T \quad (4.4)$$

satisfies the initial value

$$\vec{m}(\theta, 0) = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix}^T$$

and boundary values

$$\begin{aligned} \vec{m}(0, t) &= \begin{pmatrix} \cos(-ht) \\ \sin(-ht) \\ 0 \end{pmatrix}^T, \\ \vec{m}(2\pi, t) &= \begin{pmatrix} \cos(2\pi - ht) \\ \sin(2\pi - ht) \\ 0 \end{pmatrix}^T. \end{aligned}$$

Example 4.7. Considering the HLZ with external magnetic field $\vec{H} = (h_1, 0, 0)$ and $\alpha(t) \equiv 1$, we recast the equation as follows:

$$\begin{cases} \frac{\partial \vec{m}}{\partial t} = \vec{m} \times (\Delta \vec{m} + (h_1, 0, 0)), & \text{in } R^n \times (0, \infty), \\ \vec{m} \in S^2, & \text{in } R^n \times [0, \infty), \end{cases} \quad (4.5)$$

in which h_1 is nonzero constant.

Similarly, using Theorem 4.4, we obtain

$$\vec{m}(\theta, t) = \begin{pmatrix} 0 \\ \sin(\theta - h_1 t) \\ -\cos(\theta - h_1 t) \end{pmatrix}^T \quad (4.6)$$

that is an explicit dynamic solution of the Eq.(4.5), and satisfies the initial value

$$\vec{m}(\theta, 0) = \begin{pmatrix} 0 \\ \sin \theta \\ -\cos \theta \end{pmatrix}^T$$

and boundary values

$$\vec{m}(0, t) = \begin{pmatrix} 0 \\ \sin(-h_1 t) \\ -\cos(-h_1 t) \end{pmatrix}^T,$$

$$\vec{m}(2\pi, t) = \begin{pmatrix} 0 \\ \sin(2\pi - h_1 t) \\ -\cos(2\pi - h_1 t) \end{pmatrix}^T.$$

Remark 4.8. Investigating limit behavior of solutions for the HLZ, we nowadays face this problem that we don't find proper conditions to establish some necessary and sufficient conditions to ensure that the solution of two-dimensional Landau-Lifshitz equation under non-vanishing external magnetic field converges to the solution of the Landau-Lifshitz equation without external magnetic field if the external magnetic field tends to zero (as $\alpha(t) \rightarrow 0$). In order to solve this problem, further study needs to be finished in the future.

5 Conclusions

In the present work, we obtain some new explicit dynamic solutions of the Landau-Lifshitz equation in 2-dimensional cylindrical symmetric system that contains two cases: the generalized uniaxial anisotropic case and the external magnetic field. These explicit dynamic solutions don't equip with some properties of solutions depending only on the radius r and time t . The construction of these solutions are based on an explicit transform and an ansatz about the solution. However, many interesting problems are still unresolved here. Such as:

- (i) How is the stability of these explicit dynamic solutions in the energy space?
- (ii) Whether we can find suitable transform to construct explicit dynamic solution of the HLL, if the third component of static solution to the HLX depends only on the angle θ .

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Competing Interests

Author has declared that no competing interests exist.

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