Open Journal of
Discrete Applied Mathematics

PSR Press

Article

Spectral conditions for the Bipancyclic Bipartite graphs

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Academic Editor: Muhammad Imran
Received: 12 July 2021; Accepted: 30 July 2021; Published: 31 August 2021

Abstract: Let $G = (X, Y; E)$ be a bipartite graph with two vertex partition subsets X and Y. G is said to be balanced if $|X| = |Y|$ and G is said to be bipancyclic if it contains cycles of every even length from 4 to $|V(G)|$. In this note, we present spectral conditions for the bipancyclic bipartite graphs.

Keywords: Spectral condition; Bipancyclic Bipartite Graphs.

MSC: 05C50, 05C45.

1. Introduction

We consider only finite undirected graphs without loops or multiple edges. Notation and terminology not defined here follow that in [1]. Let $G = (V(G), E(G))$ be a graph. The graph G is said to be Hamiltonian if it contains a cycle of length $|V(G)|$. The graph G is said to be pancyclic if it contains cycles of every length from 3 to $|V(G)|$. Let $G = (V_1(G), V_2(G); E(G))$ be a bipartite graph with two vertex partition subsets $V_1(G)$ and $V_2(G)$. The bipartite graph G is said to be semiregular bipartite if all the vertices in $V_1(G)$ have the same degree and all the vertices in $V_2(G)$ have the same degree. The bipartite graph G is said to be balanced if $|V_1| = |V_2|$. Clearly, if a bipartite graph is Hamiltonian, then it must be balanced. The bipartite graph G is said to be bipancyclic if it contains cycles of every even length from 4 to $|V(G)|$. The balanced bipartite graph $G_1 = (A, B; E)$ of order $2n$ with $n \geq 4$ is defined as follows: $A = \{a_1, a_2, ..., a_n\}$, $B = \{b_1, b_2, ..., b_n\}$, and $E = \{a_i b_j : 1 \leq i \leq 2, (n-1) \leq j \leq n\} \cup \{a_i b_j : 3 \leq i \leq n, 1 \leq j \leq n\}$. Notice that G_1 is not Hamiltonian.

The eigenvalues of a graph G, denoted $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$, are defined as the eigenvalues of its adjacency matrix $A(G)$. Let $D(G)$ be a diagonal matrix such that its diagonal entries are the degrees of vertices in a graph G. The Laplacian matrix of a graph G, denoted $L(G)$, is defined as $D(G) - A(G)$, where $A(G)$ is the adjacency matrix of G. The eigenvalues $\mu_1(G) \geq \mu_2(G) \geq \cdots \geq \mu_{n-1}(G) \geq \mu_n(G) = 0$ of $L(G)$ are called the Laplacian eigenvalues of G. The second smallest Laplacian eigenvalue $\mu_{n-1}(G)$ is also called the algebraic connectivity of the graph G (see [2]). The signless Laplacian matrix of a graph G, denoted $Q(G)$, is defined as $D(G) + A(G)$, where $A(G)$ is the adjacency matrix of G. The eigenvalues $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$ of $Q(G)$ are called the signless Laplacian eigenvalues of G.

Yu *et al.*, in [3] obtained some spectral conditions for the pancyclic graphs. Motivated by the results in [3], we present spectral conditions for the bipancyclic bipartite graphs. The main results are as follows:

Theorem 1. *Let $G = (X, Y; E)$ be a connected balanced bipartite graph of order $2n$ with $n \geq 4$, e edges, and $\delta \geq 2$. If $\lambda_1 \geq \sqrt{n^2 - 2n + 4}$, then G is bipancyclic.*

Theorem 2. *Let $G = (X, Y; E)$ be a connected balanced bipartite graph of order $2n$ with $n \geq 4$, e edges, and $\delta \geq 2$. If*

$$\mu_{n-1} \geq \frac{2(n^2 - 2n + 4)}{n},$$

then G is bipancyclic.

Theorem 3. *Let $G = (X, Y; E)$ be a connected balanced bipartite graph of order $2n$ with $n \geq 4$, e edges, and $\delta \geq 2$. If*

$$q_1 \geq \frac{2(n^2 - n + 2)}{n},$$

then G is bipancyclic.

2. Lemmas

In order to prove the theorems above, we need the following results as our lemmas:
Lemma 1 is the main result in [4].

Lemma 1. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ with $n \geq 4$. Suppose that $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$, $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$, and $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. If*

$$d(x_k) \leq k < n \Longrightarrow d(y_{n-k}) \geq n - k + 1,$$

then G is bipancyclic.

Lemma 2 below follows from Proposition 2.1 in [5]:

Lemma 2. *Let G be a connected bipartite graph of order $n \geq 2$ and $e \geq 1$ edges. Then $\lambda_1 \leq \sqrt{e}$. If $\lambda_1 = \sqrt{e}$, then G is a complete bipartite graph $K_{s,t}$, where $e = st$.*

Lemma 3 below is Lemma 4.1 in [2]:

Lemma 3. *Let G be a noncomplete graph. Then $\mu_{n-1} \leq \kappa$, where κ is the vertex connectivity of G.*

Lemma 4 below is Theorem 2.9 in [6]:

Lemma 4. *Let G be a balanced bipartite graph of order $2n$ and e edges. Then $q(G) \leq \frac{e}{n} + n$.*

Lemma 5 below is Lemma 2.3 in [7]:

Lemma 5. *Let G be a connected graph. Then*

$$q_1 \leq \max\{d(u) + \frac{\sum_{v \in N(u)} d(v)}{d(u)} : u \in V\},$$

with equality holding if and only if G is either semiregular bipartite or regular.

Lemma 6. *Let $G = (X, Y; E)$ be a balanced bipartite graph of order $2n$ with $n \geq 4$, e edges, and $\delta \geq 2$. If $e \geq n^2 - 2n + 4$, then G is bipancyclic or $G = G_1$.*

Proof. Without loss of generality, we assume that $X = \{x_1, x_2, ..., x_n\}$, $Y = \{y_1, y_2, ..., y_n\}$, $d(x_1) \leq d(x_2) \leq \cdots \leq d(x_n)$, and $d(y_1) \leq d(y_2) \leq \cdots \leq d(y_n)$. Suppose G is not bipancyclic. Then Lemma 1 implies that there exists an integer k such that $1 \leq k < n$, $d(x_k) \leq k$, and $d(y_{n-k}) \leq n - k$. Thus

$$\begin{aligned} 2n^2 - 4n + 8 &\leq 2e \\ &= \sum_{i=1}^{n} d(x_i) + \sum_{i=1}^{n} d(y_i) \\ &\leq k^2 + (n-k)n + (n-k)^2 + kn \\ &= 2n^2 - 4n + 8 - (k-2)(2n - 2k - 4). \end{aligned}$$

Since $\delta \geq 2$, we have that $k \neq 1$. Therefore we have the following possible cases.
Case 1. $k = 2$.

In this case, all the inequalities in the above arguments now become equalities. Thus $d(x_1) = d(x_2) = 2$, $d(x_3) = \cdots = d(x_n) = n$, $d(y_1) = \cdots = d(y_{n-2}) = n-2$, and $d(y_{n-1}) = d(y_{n-2}) = n$. Hence $G = G_1$.

Case 2. $(2n - 2k - 4) = 0$.

In this case, we have $n = k + 2$ and all the inequalities in the above arguments now become equalities. Thus $d(x_1) = \cdots = d(x_{n-2}) = n-2$, $d(x_{n-1}) = d(x_n) = n$, $d(y_1) = d(y_2) = n-2$, and $d(y_3) = \cdots = d(y_{n-2}) = n$. Hence $G = G_1$.

Case 3. $k \geq 3$ and $2n - 2k - 4 < 0$.

In this case, we have that $n < k + 2$, namely, $n \leq k + 1$. Since $k < n$, we have $k = n - 1$. This implies that $d(y_1) \leq 1$, contradicting to the assumption of $\delta \geq 2$.

This completes the proof of Lemma 6. □

3. Proofs

Proof of Theorem 1. Let G be a graph satisfying the conditions in Theorem 1. Then we, from Lemma 2, we have

$$\sqrt{n^2 - 2n + 4} \leq \lambda_1 \leq \sqrt{e}.$$

Thus $e \geq n^2 - 2n + 4$. Therefore by Lemma 6 we have that G is bipancyclic or $G = G_1$.

If $G = G_1$, then $e = n^2 - 2n + 4$. Hence

$$\sqrt{n^2 - 2n + 4} \leq \lambda_1 \leq \sqrt{e} = \sqrt{n^2 - 2n + 4}.$$

So $\lambda_1 = \sqrt{e}$. Lemma 2 implies that G_1 is a complete bipartite graph, a contradiction.

This completes the proof of Theorem 1. □

Proof of Theorem 2. Let G be a graph satisfying the conditions in Theorem 2. Then we, from Lemma 3, we have

$$\frac{2(n^2 - 2n + 4)}{n} \leq \mu_{n-1} \leq \kappa \leq \delta \leq \frac{2e}{n}.$$

Thus $e \geq n^2 - 2n + 4$. Therefore by Lemma 6 we have that G is bipancyclic or $G = G_1$.

If $G = G_1$, then $e = n^2 - 2n + 4$. Hence

$$\frac{2(n^2 - 2n + 4)}{n} \leq \mu_{n-1} \leq \kappa \leq \delta \leq \frac{2e}{n} = \frac{2(n^2 - 2n + 4)}{n}.$$

This implies that G_1 is a regular graph, a contradiction.

This completes the proof of Theorem 2. □

Proof of Theorem 3. Let G be a graph satisfying the conditions in Theorem 3. Then we, from Lemma 4, we have

$$\frac{2(n^2 - n + 2)}{n} \leq q_1 \leq \frac{e}{n} + n.$$

Thus $e \geq n^2 - 2n + 4$. Therefore by Lemma 6 we have that G is bipancyclic or $G = G_1$.

If $G = G_1$, then $e = n^2 - 2n + 4$. Therefore

$$\frac{2(n^2 - n + 2)}{n} \leq q_1 \leq \frac{e}{n} + n = \frac{n^2 - 2n + 4}{n} + n = \frac{2(n^2 - n + 2)}{n}.$$

Hence

$$q_1 = \frac{2(n^2 - n + 2)}{n}.$$

It can be verified that

$$\max\{d(u) + \frac{\sum_{v \in N(u)} d(v)}{d(u)} : u \in V\} = \frac{2(n^2 - n + 2)}{n} = q_1.$$

Thus Lemma 5 implies that G_1 is semiregular or regular, a contradiction.

This completes the proof of Theorem 3. □

Conflicts of Interest: "The author declares no conflict of interest."

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