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Doubly Non-linear Elliptic-parabolic Equations by Rothe's Method

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Abstract

This work is devoted to study a doubly non linear elliptic-parabolic problem with quadratic gradient term by Rothe's method. We investigate the long time behavior of the solution to the discrete problem and prove the existence of compact global attractor. Our method relays on semi-discretization with respect to the time variable.

Keywords: Semi-discretization, Euler forward scheme, attractor, parabolic, elliptic, existence, uniqueness, stability.

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1 Introduction

The aim of this paper is to study a doubly non linear elliptic-parabolic equations by means of time discretization, based on the Euler forward scheme. We will approximate the parabolic problem by a sequence of elliptic problems. We prove the existence of compact global attractor. We will get our results by a semi discretization process. To this end, we investigate first existence, uniqueness and stability results for the semidiscretized problem.

We recall that the Euler forward scheme has been used by several authors while studying time discretization of nonlinear parabolic problems and we refer for example to the works [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11] and the references cited therein for some details. This scheme is usually used to prove existence of solutions as well as to compute the numerical approximations.

The problem that we consider has a quasilinear diffusion operator and a lower order term which grows quadratically in the gradient. The problems under consideration take the form

$$\begin{array}{c} \frac{\partial b(u)}{\partial t} - div(A(x)\nabla u) + h(.,t,u,\nabla u) = 0 \text{ in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \\ b\left(u(.\,,0)\right) = b\left(u_0\right) \text{ in } \Omega, \end{array} \tag{1.1}$$

and

$$\begin{array}{c} b(u) - \tau\, div(A(x)\nabla u) + \tau\, \tilde{h}(.,u,\nabla u) = 0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{array} \tag{1.2}$$

Ω Herefrom denotes an open bounded subset of $\mathbb{R}^d$, $d > 2$, with smooth boundary $\partial\Omega$. For $T > 0$, we use the following notations

$$\begin{array}{rcl} Q_T &=& \Omega\times]0,T[, \\ \Gamma_T &=& \partial\Omega\times]0,T[. \end{array}$$

$u(x,t) : Q_T \to \mathbb{R}$ is the unknown function that is sought, b is an increasing locally Lipschitz function from $\mathbb{R}$ to $\mathbb{R}$, and $A(x) = (a_{i,j}(x))$ is a matrix of $L^\infty(\Omega)$ functions $a_{i,j}(x)$ satisfying uniform ellipticity and boundedness conditions.

By a weak solution of problem (1.1) we mean a function u such that $\partial_t b(u) \in L^2(0,T;H^{-1}(\Omega))$ and satisfying

$$\int_0^T < \partial_t b(u), \varphi >_{H^{-1}(\Omega)\times H_0^1(\Omega)} + \int_{Q_T} A(x)\nabla u \nabla\varphi + \int_{Q_T} h(x,t,u,\nabla u)\varphi = 0, \tag{1.3}$$

for all $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$.

Despite recent efforts, problems (1.1) and (1.2) are in general still very poorly investigated. Let us note that the conservation law

$$\partial_t u + div f(u) = 0 \tag{1.4}$$

is a limit case of (1.1). An L^∞ entropy solution theory for the Cauchy problem for scalar conservation laws was developed by Kružkov [12] and Volpert [13]. More detailed exposition of Kružkovs theory can be found in, e.g., [14]. We also refer to [14, 15, 16] for a corresponding theory for the Dirichlet boundary value problem.

We point out that the existence of a global attractor is investigated for the following problem

$$\begin{array}{c} \frac{\partial b(u)}{\partial t} - div(A(x,u,\nabla u)) + h(x,t,u) = 0 \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \\ b\left(u(.\,,0)\right) = b\left(u_0\right) \text{ in } \Omega, \end{array} \tag{1.5}$$

(1.6)

by many authors in [1, 6, 10, 11], in those works the growth is without quadratic gradient term. When the growth is quadratic with respect to the gradient an existence result for elliptic problem was proved in [8]. The main point in this work is to explicitly include a lower order term which grows quadratically in the gradient. This does not seem to have been studied in the literature.

Many other partial differential equations are also special cases of (1.6). Let us mention the heat equation

$$\partial_t u = \Delta u, \tag{1.7}$$

and more generally elliptic-parabolic equations of the type

$$\partial_t b(u) = div\,(a(x)\nabla u), \tag{1.8}$$

where $b : \mathbb{R} \to \mathbb{R}$ is a continuous nondecreasing function and $A(x) = (a_{i,j}(x))$ is a matrix of $L^\infty(\Omega)$ functions $a_{i,j}(x)$ satisfying uniform ellipticity and boundedness conditions

$$\alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \qquad \forall \xi \in \mathbb{R}^d, \quad a.e. \quad x \in \Omega. \tag{1.9}$$

We refer to ([17], –,[30]) and the references cited therein for more information on elliptic-parabolic equations.

In this work we suppose that $h : \mathbb{R}^d \times \mathbb{R}^2 \times \mathbb{R}^d \to \mathbb{R}$ is such that:

$$h(x,t,s,\xi) = f(x,t) + b_0(s)|\xi|^2 \quad \forall\, t \in \mathbb{R}, \;\; a.e. \;\; x \in \Omega, \tag{1.10}$$

where, b_0 is an increasing locally Lipschitz function, and we can assume without loss of generality $b_0\,(0) = 0$, and

$$f \in L^2(Q_T), \;\; u_0 \in L^\infty(\Omega). \tag{1.11}$$

By using implicit Euler discretization, we discretize the problem (1.1) as follows

$$\begin{gathered} b(u^n) - \tau\, div(A(x)\nabla u^n) + \tau\tilde{h}(x,u^n,\nabla u^n) = b(u^{n-1}) \text{ in } \Omega, \\ u^n = 0 \;\text{ on } \partial\Omega, \\ b(u^0) = b(u_0) \;\text{ in } \Omega. \end{gathered} \tag{1.12}$$

A convergence proof is given for relaxation approximations and the existence of an absorbing set is obtained. We show also that all solutions are drawn, sooner or later, into a bounded set.

2 Assumptions and Main Results

In this section we introduce some notations and assumptions which will be used in the sequel. We denote by c positive constant which may vary from line to line. We define for $t \in \mathbb{R}$ the function $\psi(t)$ by

$$\psi(t) = \int_0^t b(\tau)d\tau. \tag{2.1}$$

Then the Legendre transform ψ^* of ψ is defined by

$$\psi^*(\tau) = \sup_{s\in\mathbb{R}}\{\tau s - \psi(s)\}. \tag{2.2}$$

The two main problems are the following

$$\begin{gathered} \partial_t b(u) - div(A(x)\nabla u) + h(.,t,u,\nabla u) = 0 \text{ in } \Omega\times]0,T[, \\ u = 0 \;\text{ on } \partial\Omega\times]0,T[, \\ b(u(.\,,0)) = b(u_0) \;\text{ in } \Omega, \end{gathered} \tag{2.3}$$

and

$$b(u) - \tau div(A(x)\nabla u) + \tau\tilde{h}(., u, \nabla u) = 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{2.4}$$

with

$$u_0 \in L^{\infty}(\Omega), \tag{2.5}$$

$A(x) = (a_{i,j}(x))$ is a matrix of $L^{\infty}(\Omega)$ functions $a_{i,j}(x)$ satisfying uniform ellipticity and boundedness conditions

$$\alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \qquad \forall\xi \in \mathbb{R}^d, \quad a.e. \quad x \in \Omega. \tag{2.6}$$

In order that the semidiscretized problem has unique solution we need some supplementary quadratic growth conditions, on one hand

$$|\frac{\partial h(x,t,s,\xi)}{\partial \xi}| \leq C_0(|s|)(|\xi| + \bar{b}_1(x)) \;\; a.e.\; x \in \Omega, \;\; t, s \in \mathbb{R}\;, \; \xi \in \mathbb{R}^d, \tag{2.7}$$

and

$$h(x,t,s,0) \leq C_1(|s|)\bar{b}_2(x) \;\; a.e.\; x \in \Omega, \;\; t, s \in \mathbb{R}, \tag{2.8}$$

where C_0 and C_1 are continuous functions of $|s|$, $\bar{b}_1 \in L^d(\Omega)$ and $\bar{b}_2 \in L^{\frac{d}{2}}(\Omega)$.

On an other hand, we suppose

$$|\frac{\partial h(x,t,s,\xi)}{\partial s}| \geq \alpha_0 \;\; a.e.\; x \in \Omega, \;\; t, s \in \mathbb{R}\;, \; \xi \in \mathbb{R}^d \text{ and } \alpha_0 > 0. \tag{2.9}$$

Let us note that in [5] the same assumptions where considered in order to prove the maximum principle for solutions in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$. This implies in particular the uniqueness of the solution of the semidiscretized problem in $H_0^1(\Omega) \cap L^{\infty}(\Omega)$.

Let us recall this version of discrete uniform Gronwall's Lemma [30]

Lemma 2.1. *Let $(y^n)_{n\geq 0}$ and $(h^n)_{n\geq 0}$ be to sequences of real numbers, not necessarily positive, satisfying*

$$y^n \leq y^{n-1} + \tau h_n$$

and there exists a positive integer n_0 such that for all $n_1 \geq n_0$ and $N > 0$

$$\tau \sum_{n=n_1}^{n_1+N} h_n \leq l_1 \text{ and } \tau \sum_{n=n_1}^{n_1+N} y^n \leq l_2,$$

for some positive real numbers l_1 and l_2 that do not depend on n_1, then for all $n_1 \geq n_0$

$$y^{n_1+N} \leq \frac{l_2}{\tau N} + l_1.$$

To study existence and regularity by semi discretization in time, we consider the following problems

$$\begin{gathered} b(u^n) - \tau\, div(A(x)\nabla u^n) + \tau\tilde{h}(x, u^n, \nabla u^n) = b(u^{n-1}) \text{ in } \Omega, \\ u^n = 0 \text{ on } \partial\Omega, \\ u^0 = u_0 \text{ in } \Omega, \end{gathered} \tag{2.10}$$

where $n = 1, ..., N$, $\tau N = T$, $0 < \tau < 1$,

$$\tilde{h}(x, u^n, \nabla u^n) = h(x, n\tau, u^n, \nabla u^n). \tag{2.11}$$

By a weak solution of problems (2.10) we mean a sequence of functions $(u^n)_{0\leq n\leq N}$, such that $b(u^0) = b(u_0)$ and u^n defined by induction as a weak solution of the following problem

$$\begin{gathered} b(u) - \tau\, div(A(x)\nabla u) + \tau\tilde{h}(x, u, \nabla u) = b(u^{n-1}) \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \\ u^0 = u_0 \text{ in } \Omega. \end{gathered} \tag{2.12}$$

First, we state the following stability estimation results.

Theorem 2.2. *There exists a constant $c(f, u_0)$ independent on N , such that for any $n = 1, ..., N$ on has*

$$||b(u^n)||_\infty \leq c(f, u_0), \tag{2.13}$$

$$\sum_{i=1}^{n} ||b(u^i) - b(u^{i-1})||_2^2 \leq c(f, u_0), \tag{2.14}$$

$$\int_\Omega \psi^*(b(u^n))dx + \tau \sum_{i=1}^{n} ||u^i||_{1,2}^2 \leq c(f, u_0). \tag{2.15}$$

□

Next, we prove the following main result.

Theorem 2.3. *There exist a compact attractor A that attracts all the solutions u^n of the discrete problem in the sense that*

$$\lim_{n\to+\infty} dist(A, u^n) = 0,$$

where,

$$dist(x, M) = \inf_{y\in M} d(x, y).$$

3 Semidiscretized Problem

We shall study the following elliptic problem

$$\begin{cases} b(u^n) - \tau div(A(x)\nabla u^n) + \tau\tilde{h}(., u^n, \nabla u^n) = b(u^{n-1}) & \text{in } \Omega, \\ u^n = 0 & \text{on } \partial\Omega, \\ b\left(u^0\right) = b(u_0) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where $n = 1, ..., N$, $\tau N = T$, $0 < \tau < 1$ and

$$f_n\left(.\right) = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} f(s, .)ds,\ t^n = n\tau. \tag{3.2}$$

For $n = 1$, we consider the problem

$$\begin{gathered} b(u^1) - \tau div(A(x)\nabla u^1) + \tau\tilde{h}(., u^1, \nabla u^1) = b(u_0) \text{ in } \Omega, \\ u^1 = 0 \text{ on } \partial\Omega, \\ u^0 = u_0, \end{gathered} \tag{3.3}$$

such that the function $\tilde{h}(., ., .)$ satisfying the following hypothesis

$$\begin{aligned} |\tilde{h}(., u^1, \nabla u^1)| &\leq \overline{f} - \tau b(u^1)|\nabla u^1|^2 + k. \\ \overline{f} &= f_1 + b(u_0). \end{aligned}$$

Let us note that $\overline{f} = f_1 + b(u_0) \in L^2(\Omega)$. We point out that existence results of bounded solutions have been obtained in [2, 19, 20]. In general, it is well known that one can expect an L^∞

solution if $f \in W^{-1,q}(\Omega)$, $q > d$. Bounded Solutions are also obtained in [8]. Then problem 3.3 has one bounded solution, by induction, we deduce that for any $n = 2, ..., N$ problem 3.1 has one solution.

Let us now prove Theorem 2.2. Substituting φ by $b(u^i)|b(u^i)|^q$ and n by i in (3.1) and using Hölder's inequality, one has

$$||b(u^i)||^{q+2}_{L^{q+2}} \leq ||b(u^i)||^{q+1}_{L^{q+2}}||b(u^{i-1})||_{L^{q+2}} + \tau c||b(u^i)||^{q+1}_{L^{q+1}}.$$

It follows that

$$||b(u^i)||_{L^{q+2}} \leq ||b(u^{i-1})||_{L^{q+2}} + c\tau.$$

By induction, we obtain

$$||b(u^n)||_{L^{q+2}} \leq ||b(u_0)||_{L^{q+2}} + cT,$$

letting q go to infinity we obtain

$$||b(u^n)||_{L^\infty(\Omega)} \leq c(f, u_0, T). \tag{3.4}$$

Next, we substitute φ by $b(u^i)$ and n by i in the weak formulation of (3.1). One has

$$\int_\Omega (b(u^i) - b(u^{i-1}))b(u^i) + \tau\alpha \int_\Omega |\nabla u^i|^2 \leq \tau \int_\Omega f_i b(u^i) + \int_\Omega b(u^i). \tag{3.5}$$

Using the fact that

$$2a(a-b) = a^2 - b^2 + (a-b)^2,$$

we obtain

$$\tau\alpha||\nabla u^i||^2_2 + ||b(u^i)||^2_{L^2} - ||b(u^{i-1})||^2_{L^2} + ||b(u^i) - b(u^{i-1})||^2_{L^2} \leq ||b(u^i)||_{L^1} c(\tau + 1).$$

Which yields that

$$\tau\alpha||\nabla u^i||^2_2 + ||b(u^n)||^2_{L^2} + \sum_{i=1}^n ||b(u^i) - b(u^{i-1})||^2_{L^2} \leq ||b(u^0)||^2_{L^2} + cT\sum_{i=1}^n ||b(u^i)||_{L^1}.$$

This implies that

$$\sum_{i=1}^n ||b(u^i) - b(u^{i-1})||^2_{L^2} \leq c(f, u_0, T), \tag{3.6}$$

and

$$||\nabla u^i||^2_2 \leq c(f, u_0, T), \tag{3.7}$$

Finally, we take $\varphi = u_i$ and we substitute n by i in the weak formulation of (3.1). We obtain

$$\int_\Omega (b(u^i) - b(u^{i-1}))u^i + \tau\alpha \int_\Omega |\nabla u^i|^2 \leq \tau \int_\Omega f_i u^i + \int_\Omega u^i, \tag{3.8}$$

$$\int_\Omega \psi^*(b(u^i)) - \int_\Omega \psi^*(b(u^{i-1})) + \tau\alpha \int_\Omega |\nabla u^i|^2 \leq \tau \int_\Omega f_i u^i + \int_\Omega u^i,$$

$$\int_\Omega \psi^*(b(u^i)) - \int_\Omega \psi^*(b(u^{i-1})) + \tau\alpha||u^i||^2_{W^{1,2}} \leq c\tau||u^i||_{L^1}.$$

Then summing from $i = 1$ to n, we obtain

$$\int_\Omega \psi^*(b(u^n)) + \alpha\tau \sum_{i=1}^n ||u^i||^2_{W^{1,2}} \leq c\tau \sum_{i=1}^n ||u^i||_{L^1} + \int_\Omega \psi^*(b(u_0)),$$
$$\leq c(f, u_0, T).$$

Then

$$\int_\Omega \psi^*(b(u^n)) + \alpha\tau \sum_{i=1}^n ||u^i||^2_{W^{1,2}} \leq c(f, u_0, T). \tag{3.9}$$

4 Compact Attractor by Discrete Dynamical System

We consider the following Rothe function u^N defined by

$$\begin{aligned} b(u^N(0)) &= b(u_0), \\ b(u^N(t)) &= b(u^{n-1}) + (b(u^n) - b(u^{n-1}))(\tfrac{t-t^{n-1}}{\tau}), \text{for any } t \in]t^{n-1}, t^n],\ n = 1, ..., N, \end{aligned} \tag{4.1}$$

and the piecewise constant function $\bar{u}^N$ defined by

$$\begin{aligned} b(\bar{u}^N\ (0)) &= b(u_0), \\ b(\bar{u}^N\ (t)) &= b(u^n),\ \text{for any } t \in]t^{n-1}, t^n],\ n = 1, ..., N. \end{aligned}$$

Since the problem (2.10) has a unique solution $(u^n)_{0\leq n\leq N}$ then the functions $b(u^N)$ and $b(\bar{u}^N)$ are uniquely defined and by construction, we have for any $t \in]t^{n-1},\ t^n]$ and $n = 1, ..., N$, that

$$\begin{aligned} \frac{\partial b(u^N(t))}{\partial t} &= \frac{b(u^n) - b(u^{n-1})}{\tau}, \\ b(\bar{u}^N(t)) - b(u^N(t)) &= (b(u^n) - b(u^{n-1}))\frac{t^n - t}{\tau}. \end{aligned}$$

By using the stability results of Theorem 2.2, we deduce the following a priori estimates concerning the function $b(u^N)$ and the function $b(\bar{u}^N)$.

Lemma 4.1. *There exists a constant $c(f, u_0, T)$ independent of N such that for all $N \in \mathbb{N}$, we have*

$$||b(u^N) - b(\bar{u}^N)||^2_{L_2(Q_T)} \leq \frac{1}{N}c(f, u_0, T), \tag{4.2}$$

$$||b(u^N)||_{L_2(Q_T)} \leq c(f, u_0, T), \tag{4.3}$$

$$||b(\bar{u}^N)||_{L_2(Q_T)} \leq c(f, u_0, T), \tag{4.4}$$

$$||b(\bar{u}^N)||^2_{1,2} \leq c(f, u_0, T). \tag{4.5}$$

$$||\frac{\partial b(u^N)}{\partial t}||_{L_2(0,T;H^{-1})} \leq c(f, u_0, T), \tag{4.6}$$

Proof. To prove (4.2) we notice that on has

$$\begin{aligned} ||b(u^N) - b(\bar{u}^N)||^2_{L_2(Q_T)} &= \sum_{n=1}^{N}\int_{t^{n-1}}^{t_n} ||b(u^n) - b(u^{n-1})||^2_2 \left(\frac{t^n - t}{\tau}\right)^2 dt \\ &= \frac{\tau}{3}\sum_{n=1}^{N} ||b(u^n) - b(u^{n-1})||^2_2. \end{aligned}$$

From 3.6 on has

$$||b(u^N) - b(\bar{u}^N)||^2_{L_2(Q_T)} \leq \frac{1}{N}c(f, u_0, T). \tag{4.7}$$

And, by the same way as (4.2) we prove (4.3), (4.4) and (4.5).

To prove (4.6) we consider the set $A = \{\varphi \in H_0^1(\Omega) : ||\varphi|| \leq 1\}$, then we have

$$\begin{aligned} ||\frac{\partial b(u^N)}{\partial t}||_{L_2(0,T;H^{-1})} &= \int_0^T \sup_{\varphi \in A} < \frac{\partial b(u^N)}{\partial t}, \varphi > dt \\ &= \sum_{i=1}^{N} \sup_{\varphi \in A} < \frac{b(u^i) - b(u^{i-1})}{\tau}, \varphi > \\ &\leq \sum_{i=1}^{N} \sup_{\varphi \in A} (\tau\beta \int_\Omega |\nabla u^i||\nabla\varphi| + \tau \\ &\int_\Omega |b(u^i)\varphi|\nabla u^i|^2\varphi| + \tau \int |f_i\varphi|) \\ &\leq \tau\beta \sum_{i=1}^{N} ||\nabla u^i||_2 + \tau \sum_{i=1}^{N} ||b(u^i)||^2 + c_2. \end{aligned}$$

From (3.7) on has

$$||\frac{\partial b(u^N)}{\partial t}||_{L_2(0,T;H^{-1})} \leq c(f, u_0, T). \tag{4.8}$$

□

From the estimates of the previous lemma, we deduce that there exists a function $u \in H_0^1(Q_T)$ such that

$$b(u^N) \to b(u) \text{ in } L^2(Q_T), \tag{4.9}$$

$$b(\bar{u}^N) \to b(u) \text{ in } L^2(Q_T), \tag{4.10}$$

$$\frac{\partial b(u^N)}{\partial t} \to \frac{\partial b(u)}{\partial t} \text{ weakly in } L^2(0,T;H^{-1}(\Omega)), \tag{4.11}$$

$$\nabla\bar{u}^N \to \nabla u \text{ weakly in } L^2(Q_T)^N. \tag{4.12}$$

$$b(\bar{u}^N)|\nabla\bar{u}^N|^2 \to b(u)|\nabla u|^2 \text{ weakly in } L^2(Q_T). \tag{4.13}$$

By definition of $\left(u^N\right)_{N\in\mathbb{N}}$, we have $b(u^N(0)) = b(u^0) = b(u_0)$ for all $N \in \mathbb{N}$. Then $b(u(0,.)) = b(u_0)$.
Taking a test function $\varphi \in L^2(0,T;H_0^1(\Omega)) \cap L^\infty(Q_T)$ in the weak formulation we obtain

$$\int_0^T < \frac{\partial b(u^N)}{\partial t}, \varphi > + \int_{Q_T} A(x)\nabla\bar{u}^N\nabla\varphi + \int_{Q_T} \bar{h}(x,t,\bar{u}^N,\nabla\bar{u}^N)\varphi = 0, \tag{4.14}$$

where

$$|\bar{h}(x,t,\bar{u}^N,\nabla\bar{u}^N)| \leq f_N(t,x) - b(\bar{u}^N)|\nabla\bar{u}^N|^2 + k$$

and

$$f_N(t,x) = f_n(x) \text{ for any } t \in]t^{n-1}, t^n], n = 1, ..., N.$$

Tending N to infinity, by standard argument, we obtain the desired result.

Let us now define the map S_τ by

$$S_\tau u^{n-1} = u^n. \tag{4.15}$$

Then

$$S_\tau^n u^0 = u^n. \tag{4.16}$$

In order that the nonlinear map S_τ satisfies the properties of the semi groups:

$$S_\tau^{n+p} = S_\tau^n \circ S_\tau^p, \tag{4.17}$$

we need (3.1) to be autonomous. For this purpose we further need to assume that the nonlinear function h is independent of the time, that is $h(t, x, s, \xi) = h(x, s, \xi)$.

We are now in stage to prove that the discrete problem has a compact attractor that attracts all the solutions in the sense that

$$\lim_{n\to+\infty} dist(A, S_\tau^n u^0) = 0.$$

To this end, we prove that there exists an absorbing ball B in $L^\infty(\Omega) \cap H_0^1(\Omega)$ independent on τ.

Let us consider

$$\tilde{H}_b(u) = \int_0^u \tilde{h}(x, s, \xi) + c\, b(s)ds, \tag{4.18}$$

and multiplying the discrete problem by $u^n - u^{n-1}$ we obtain

$$\int_\Omega (b(u^n) - b(u^{n-1}))(u^n - u^{n-1}) + \tau \int_\Omega A(x)\nabla u^n \nabla(u^n - u^{n-1}) + \tau \int_\Omega \tilde{h}(x, u^n, \nabla u^n)(u^n - u^{n-1}) = 0.$$

From to the growth condition on $\tilde{h}$ and b we have $\tilde{H}_b(u)$ is convex then we obtain

$$\tilde{H}_b(u)(u - v) \geq \tilde{H}_b(u) - \tilde{H}_b(v).$$

It follows that

$$\int_\Omega (\tilde{h}(x, u^n, \nabla u^n) + cb(u^n))(u^n - u^{n-1})dx \geq \int_\Omega \tilde{H}_b(u^n) - \tilde{H}_b\left(u^{n-1}\right)dx. \tag{4.19}$$

Then, we obtain from the equality

$$\int_\Omega \tilde{h}(x, u^n, \nabla u^n)(u^n - u^{n-1})dx = \int_\Omega (\tilde{h}(x, u^n, \nabla u^n) + cb(u^n))(u^n - u^{n-1}) - c\int_\Omega b(u^n)(u^n - u^{n-1})dx,$$

that

$$\int_\Omega \tilde{h}(x, u^n, \nabla u^n)(u^n - u^{n-1})dx \geq \int_\Omega \tilde{H}_b(u^n) - \tilde{H}_b\left(u^{n-1}\right)dx - c\int_\Omega b(u^n)(u^n - u^{n-1})dx. \tag{4.20}$$

By using the remark

$$\int_\Omega b(u^n)(u^n - u^{n-1})dx = \int_\Omega (b(u^n) - b(u^{n-1}))(u^n - u^{n-1}) + \int_\Omega b(u^{n-1})(u^n - u^{n-1})dx,$$

and hypothesis (2.6) on $A(x)$, we obtain

$$\int_\Omega \tilde{H}_b(u^n)dx + c||u^n||^2_{H_0^1(\Omega)} \leq c||u^{n-1}||^2_{H_0^1(\Omega)} + \int_\Omega \tilde{H}_b\left(u^{n-1}\right)dx + c\int_\Omega b(u^{n-1})(u^n - u^{n-1})dx. \tag{4.21}$$

Let us now consider

$$\tilde{H}(u) = \int_0^u \tilde{h}(x, s, \xi)\, ds.$$

Then

$$\int_\Omega \tilde{H}_b(u)dx = \int_\Omega \tilde{H}(u)dx + c\int_\Omega \psi(u)dx.$$

Using the fact that

$$\int_\Omega b(u^{n-1})(u^n - u^{n-1})dx \leq \int_\Omega \psi(u^n) - \psi(u^{n-1})dx,$$

we obtain

$$c\int_\Omega \tilde{H}\left(u^n\right)dx + ||u^n||^2_{H^1_0(\Omega)} \le c\int_\Omega \tilde{H}\left(u^{n-1}\right)dx + ||u^{n-1}||^2_{H^1_0(\Omega)}. \tag{4.22}$$

We denote the left hand side by y^n hence the right one is y^{n-1}. Using the stability results and the discrete uniform Gronwall's lemma, here $h_n = 0$, we get an integer $m > 0$ such that

$$c\int_\Omega \tilde{H}\left(u^n\right)dx + ||u^n||^2_{H^1_0(\Omega)} \le c \qquad \text{for all } n \ge m. \tag{4.23}$$

Using the fact that b is invertible and by a repeated application of Theorem 2.2 we find an integer $m^{'}$ such that

$$u^n \in L^\infty(\Omega) \text{ for all } n \ge m^{'}$$

$$||u^{m^{'}}||_{L^\infty(\Omega)} \le c(m^{'}).$$

Then, we find an increasing sequence $(\beta(m))_{m\ge1}$ such that

$$\beta(m) \ge 2,\ \frac{1}{\beta(m+1)} = \frac{1}{\beta(m)} - \frac{1}{d}$$

and

$$||u^m||_{\beta(m)} \le \frac{c(m)}{\tau^{\beta+\beta^2+\ldots+\beta^m}}(||u_0||_2^{\beta(m)}+1).$$

We stop the iteration on m once we have $\beta(m-1) > \frac{d}{2}$. Indeed $L^{\frac{d}{2}+\epsilon}(\Omega) \subset W^{-1,r}(\Omega)$ for $r > d$ and for all $\epsilon > 0$. Then $m^{'}$ will be the first integer such that $\beta(n(d)-1) > \frac{d}{2}$. By induction, we obtain $u^n \in L^\infty(\Omega)$.

Therefore, from (4.23) we get

$$||u^n||_{H^1_0(\Omega)} \le c \qquad \text{for all } n \ge m^{'}. \tag{4.24}$$

We conclude now that the semidiscretized problem has an absorbing set B in $L^\infty(\Omega) \cap H^1_0(\Omega)$. Setting

$$A = w(B) = \bigcap_{k\ge0}\overline{\bigcup_{n\ge k} S^n_\tau(B_\tau)}$$

and applying Theorem 1.1 of [30] we therefor get A is a compact subset that attracts all the solutions in the sense that

$$dist(A, S^n_\tau u^0) \underset{n\to+\infty}{\to} 0.$$

5 Conclusion

Note that the above result is obtained for the case of bounded domains when the external forcing term f is in $L^2(Q_T)$ and u_0 in $L^\infty(\Omega)$. This can be obtained with the appropriate *a priori estimate*, using the time analyticity of the solutions, which gives a bound on $|u_t|$ (see, e.g., [16,17]). Thus, the only novelty here is for the case of lower order term which grows quadratically in the gradient. In this paper, we give a positive answer to this question.

Competing Interests

The authors declare that no competing interests exist.

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