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Solution Procedure for Systems of Partial Differential-Algebraic Equations by NHPM and Laplace-Padé Resummation

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Abstract

In this paper, we propose an efficient modification of a New Homotopy Perturbation Method (NHPM) to obtain approximate and exact analytical solutions of Partial Differential-Algebraic Equations (PDAEs). The NHPM is first applied to the PDAE to obtain the exact solution in convergent series form. To improve the solution obtained from NHPM's truncated series, a post-treatment combining Laplace transform and Padé approximant is proposed. This modified Laplace-Padé new homotopy perturbation method is shown to be effective and greatly improves NHPM's truncated series solutions in convergence rate, and often leads to the exact solution. Two problems are solved to demonstrate the efficiency of the method; the first one is a nonlinear index-one system with an integral term and the second one is a linear index-three system with variable coefficients.

Keywords: Partial differential-algebraic equations, Homotopy perturbation method, Laplace transform, Padé approximant, Analytical solutions, Resummation methods.

2010 Mathematics Subject Classification: 34L30

1 Introduction

Many complex technical systems throughout science and engineering are easily modeled by Partial Differential-Algebraic Equations (PDAEs). This type of equations arise in nanoelectronics [1], electrical networks [2], [3], [4], mechanical systems [5] and many other applications [6], [7], [8], [9], [10], [11], [12], [13], [14].

In recent years, PDAEs have received much attention because of their wide applications. The convergence of Runge-Kutta method for linear PDAEs was investigated in [15]. The study of indices and the numerical solution of linear PDAEs with constant coefficients is given in [16], [17], [18], [19]. In the literature, one can find various index definitions for PDAEs [20], [21], [22], [23]. But, the most used index is the differentiation index. It is defined as the minimum number of times that all or part of the PDAE must be differentiated with respect to time in order to obtain the time derivative of the solution as a continuous function of the solution and its space derivatives [20].

Like differential-algebraic equations (DAEs), higher-index PDAEs (differentiation index greater than one) are known to be difficult to solve numerically. Usually, these problems are first transformed to index-one systems before applying a numerical integration. This procedure is known as an index-reduction. A reduction of the index can be expensive or may change the solution properties of the original problem. Often, to treat PDAEs the method of lines is used and the PDAE is discretized in the space direction to obtain a set of DAEs [6], [10], [24]. The resulting DAE is then transformed to a lower-index system that can be easily integrated by standard DAE numerical methods [25]. This approach may be, however, inappropriate for higher-index PDAEs with multi-dimensional solutions since the index-reduction can be very expensive. Because, most of the problems arising from real-life applications are higher-index PDAEs, new techniques are needed to solve these problems efficiently.

Modern methods like homotopy perturbation method (HPM) [26], [27], [28], [29], homotopy analysis method (HAM) [30], [31], [32], variational iteration method (VIM) [33], generalized homotopy method [34], among others, are powerful tools to approximate nonlinear and linear problems. Nevertheless, at present, the Homotopy Perturbation Method (HPM) introduced by Ji Huan He [26], [27] is one among the most employed analytical methods in science and engineering to solve nonlinear/linear problems. This application includes problems like: convolution product nonlinearities [28], oscillators with discontinuities [35], nonlinear wave equations [36], Volterra's integro-differential equation [37], second-order BVPs [38], differential-algebraic equations [39, 40], stiff systems [41], neutral functional-differential equations with proportional delays [42], partial differential equations [43], electro-statically actuated microbeam [44], [45], fuzzy linear systems [46], linear programming [47], bifurcation of nonlinear problems [48] and boundary value problems [49], among many others. The HPM is a combination of the classical perturbation method and the homotopy technique. The HPM solution is considered as the sum of an infinite series which in most cases converges rapidly to the exact solution. Usually, only a few terms of the series solution are enough to achieve a high degree of accuracy.

On one side, DAEs problems are treated by using: Padé series [50], Adomian decomposition method [51], homotopy analysis method [52]. On the other side, PDAEs problems are solved by using: Variational Methods [53], Modified Homotopy Perturbation Method [54], variational iteration method [55], Differential Transform Method [56]. Moreover, in [39], some higher-index DAEs were solved by transforming them first to index-one systems before applying the HPM. Furthermore, the solution series involves noise terms which affect the performance of the method. Recently, a New Homotopy Perturbation Method (NHPM) was developed in [57], [58], [41] to solve different types of problems. This method was successfully used by Salehi [40] to solve DAEs. Additionally, Laplace-Padé post-treatment is shown to be an effective tool [59], [60], [61], [62], [63], [64], [65], [66] to increase accuracy and convergence of power series solutions obtained by HPM.

In this paper, we propose a hybrid method which combines the NHPM, Laplace transform (LT) and Padé approximant (PA) [67] to solve PDAEs analytically. Solutions of PDAEs are first obtained in convergent series forms in only one iteration using the NHPM. To improve the solutions obtained from NHPM's truncated series, we apply LT then convert the transformed series into a meromorphic function by forming its PA, and finally we take the inverse LT of the PA to obtain the modified analytic solution. This modified Laplace-Padé new homotopy perturbation method (LPNHPM) greatly improves NHPM's truncated series solutions in convergence rate, and often leads to the exact solution. Because higher-index PDAEs are difficult to solve numerically, we choose two examples of PDAEs with known exact solutions to demonstrate the efficiency of the proposed method. Additionally, the LPNHPM does not require any index-reduction to solve higher-index PDAEs and the solutions do not generate noise terms which may reduce the efficiency of the method.

The rest of this paper is organized as follows. In the next section, we illustrate the basic idea of the NHPM. In section 3, we briefly review the Padé approximant. In section 4, we give the basic concept of the Laplace-Padé resummation method. In section 5, we describe our procedure to solve PDAEs using the NHPM. In section 6, we apply the technique discussed in section 5 along with the Laplace-Padé post-treatment to solve a nonlinear index-one PDAE and a linear index-three PDAE. In section 7, we give a brief discussion. Finally, a conclusion is drawn in the last section.

2 Description of the NHPM

In this section, we illustrate the basic idea of the NHPM [41]. For this, we consider the following nonlinear first order system of ordinary differential equations

$$\frac{du}{dt} + N(u) - f(t) = 0, \ t \in [0, T], \tag{1}$$

supplied with the vector initial condition

$$u(0) = g_0, \tag{2}$$

where N is a nonlinear function, $f(t)$ is a known analytical function on $[0, T]$ and g_0 is a constant. We assume that a solution to initial-value problem (1)-(2) exists, is unique and sufficiently smooth. Note here that the method we are proposing can be applied to higher order systems of ordinary differential equations. In this case, initial condition (2) will be then on the solution and it time derivatives.

According to the HPM [27], we construct for (1) a homotopy $v(t) := v(t, p) : [0, T] \times [0, 1] \to \mathbb{R}$ that satisfies

$$\frac{dv}{dt} - \frac{du_0}{dt} + p\frac{du_0}{dt} + p\Big(N(v) - f(t)\Big) = 0, \tag{3}$$

and $v(0) = g_0$, where $p \in [0, 1]$ is an embedding parameter and u_0 is an initial approximation for the solution of (1) that satisfies $u_0(0) = g_0$. For the HPM, the initial approximation u_0 can be taken for example as the solution of the linear equation $\frac{du}{dt} = f(t)$. For the NHPM we are going to discuss, the choice of the initial approximation u_0 will be given later in this section.

Equation (3) can be written as

$$\frac{dv}{dt} = \frac{du_0}{dt} + p\Big(f(t) - \frac{du_0}{dt} - N(v)\Big). \tag{4}$$

According to the HPM, the solution v of (3) and (4) is assumed to have the form

$$v = v_0 + pv_1 + p^2 v_2 + \ldots, \tag{5}$$

where v_n, $n = 0, 1, 2, \ldots$ are unknown functions to be determined by the HPM scheme.

Setting $p = 1$ in (5), an approximate solution of equation (1) is obtained by

$$u = v_0 + v_1 + v_2 + \ldots \tag{6}$$

Integrating (4) once with respect to t and using initial conditions (2) and $v(0) = g_0$ yield

$$v(t) = u_0(t) - p\int_0^t \Big(\frac{du_0(\tau)}{d\tau} + N(v(\tau)) - f(\tau)\Big)d\tau. \tag{7}$$

Then substituting (5) into (7), we get

$$v_0 + pv_1 + p^2 v_2 + \ldots = u_0(t) - p\int_0^t \Big(\frac{du_0(\tau)}{d\tau} + N(v_0 + pv_1 + p^2 v_2 + \ldots) - f(\tau)\Big)d\tau. \tag{8}$$

Expanding the nonlinear term N in the integrand in terms of Adomian operators, we get

$$N(v_0 + pv_1 + p^2 v_2 + \ldots) = \sum_{i=0}^{\infty} N_i(v_0, \ldots, v_i)p^i, \tag{9}$$

where

$$N_0 = N_0(v_0),\ N_1 = N'(v_0)v_1,\ N_2 = N'(v_0)v_2 + N''(v_0)\frac{v_1^2}{2}, \ldots \tag{10}$$

Equating the coefficients of like powers of p in (8), we obtain the following set of equations
p^0:

$$v_0(t) = u_0(t),$$

p^1:

$$v_1(t) = g_0 - u_0(t) - \int_0^t \Big(N_0(v_0(\tau)) - f(\tau)\Big)d\tau, \tag{11}$$

p^2:

$$v_2(t) = -\int_0^t N_1(v_0, v_1)d\tau, \tag{12}$$

p^i:

$$v_i(t) = -\int_0^t N_{i-1}(v_0, v_1, \ldots, v_{i-1})d\tau,\ i = 3, 4, \ldots \tag{13}$$

To determine the solution using the HPM, one has to find the solution components v_i, $i = 0, 1, 2, \ldots$ In some cases, one is faced with large and complex computations. Additionally, the solution may involve the computation of many noise terms which affect the performance of the method [39].

To circumvent these difficulties, we propose to use the NHPM introduced in [41] which assumes the initial approximation to have the form

$$u_0(t) = \sum_{n=0}^{\infty} \alpha_n t^n, \tag{14}$$

where α_n, $n = 0, 1, 2, \ldots$ are unknown coefficients to be determined from the following system

$$v_1(t) = g_0 - u_0(t) - \int_0^t \Big(N_0(v_0(\tau)) - f(\tau)\Big)d\tau = 0. \tag{15}$$

When $v_1(t) = 0$ is satisfied, then (10) and (12) lead to $v_2(t) = 0$ for all t. In a similar manner, by using (10) and (13), we can show that $v_i(t) = 0$, for $i = 3, 4, \ldots$

Now to determine the unknown coefficients α_i, $i = 0, 1, 2, \ldots$, we substitute (14) into (15) then solve the resulting system for the coefficients α_i. Finally, we use equation (6) to obtain the exact solution u as series

$$u(t) = u_0(t) = \sum_{n=0}^{\infty} \alpha_n t^n. \tag{16}$$

3 Padé Approximant

Given an analytical function $u(t)$ with Maclaurin's expansion

$$u(t) = \sum_{n=0}^{\infty} u_n t^n,\ 0 \le t \le T. \tag{17}$$

The Padé approximant to $u(t)$ of order $[L, M]$ which we denote by $[L/M]_u(t)$ is defined by [67]

$$[L/M]_u(t) = \frac{p_0 + p_1 t + \ldots + p_L t^L}{1 + q_1 t + \ldots + q_M t^M}, \tag{18}$$

where we considered $q_0 = 1$, and the numerator and denominator have no common factors.

The numerator and the denominator in (18) are constructed so that $u(t)$ and $[L/M]_u(t)$ and their derivatives agree at $t = 0$ up to $L + M$. That is

$$u(t) - [L/M]_u(t) = O\left(t^{L+M+1}\right). \tag{19}$$

From (19), we have

$$u(t)\left(\sum_{i=1}^{M} q_i t^i\right) - \left(\sum_{i=0}^{L} p_i t^i\right) = O\left(t^{L+M+1}\right). \tag{20}$$

From (20), we get the following systems

$$\begin{cases} u_L q_1 + \ldots + u_{L-M+1} q_M = -u_{L+1} \\ u_{L+1} q_1 + \ldots + u_{L-M+2} q_M = -u_{L+2} \\ \vdots \\ u_{L+M-1} q_1 + \ldots + u_L q_M = -u_{L+M}, \end{cases} \tag{21}$$

and

$$\begin{cases} p_0 = u_0 \\ p_1 = u_1 + u_0 q_1 \\ \vdots \\ p_L = u_L + u_{L-1} q_1 + \ldots + u_0 q_L. \end{cases} \tag{22}$$

From (21), we calculate first all the coefficients q_i, $1 \leq i \leq M$. Then, we determine the coefficient p_i, $0 \leq i \leq L$ from (22).

Note that for a fixed value of $L + M + 1$, the error (19) is smallest when the numerator and denominator of (18) have the same degree or when the numerator has degree one higher than the denominator.

4 Laplace-Padé Post-treatment

Several analytical approximation methods like HPM provide solutions as power series. Nevertheless, in some situations the truncated series have limited domain of convergence. To enlarge the domain of convergence, many authors used the Laplace-Padé resummation method [40], [41], [42], [43], [44], [45], [46], [47]. This method can be summarized in the following steps:

1) Apply Laplace transform to the truncated series solution, then substitute s by $1/t$ in the resulting expression.

2) Convert the transformed series into a meromorphic function by forming its Padé approximation of order [L/M], then substitute t by $1/s$ in the resulting expression.

3) Finally, apply the inverse Laplace s-transform to obtain the modified approximate solution.

5 Solution of PDAES with LPNHPM

In this section, we will show how to apply the modified NHPM to solve PDAEs. Since many application problems in science and engineering are modelled by semi-explicit PDAEs, we consider the following class of PDAEs

$$u_{1t} = \phi\left(u, u_x, u_{xx}\right), \tag{23}$$

$$0 = \psi\left(u, u_x, u_{xx}\right),\ (t, x) \in (0, T) \times (a, b), \tag{24}$$

where u_k: $[0, T] \times [a, b] \to \mathbb{R}^{m_k}$, $k = 1$, 2, and $b > a$.

System (23)-(24) is subject to the initial condition

$$u_1\left(0, x\right) = g\left(x\right),\ a \le x \le b, \tag{25}$$

and some suitable boundary conditions

$$B\left(u\left(t, a\right), u\left(t, b\right), u_x\left(t, a\right), u_x\left(t, b\right)\right) = 0,\ 0 \le t \le T, \tag{26}$$

where $g\left(x\right)$ is a given function.

We assume that a solution to initial boundary-value problem (23)-(26) exists, is unique and sufficiently smooth.

To simplify the exposition of the method, we integrate first equation (23) with respect to time and use the initial condition (25) to obtain

$$u_1\left(t, x\right) - g\left(x\right) - \int_0^t \phi\left(u, u_x, u_{xx}\right) dt = 0. \tag{27}$$

According to the HPM, we construct the following homotopy for equations (24) and (27)

$$(1 - p)\left(v_1 - u_{1,0}\right) + p\left(v_1 - g\left(x\right) - \int_0^t \phi\left(v\right) dt\right) = 0, \tag{28}$$

$$(1 - p)\left(v_2 - u_{2,0}\right) + p\psi\left(v\right) = 0, \tag{29}$$

where $\phi\left(v\right) := \phi\left(v, v_x, v_{xx}\right)$, $\psi\left(v\right) := \psi\left(v, v_x, v_{xx}\right)$, and $p \in [0, 1]$ is an embedding parameter, $u_{1,0}$ and $u_{2,0}$ are the initial approximations for u_1 and u_2 respectively, and $v = (v_1, v_2)$ is an unknown function on the independent variables t, x, p. Note that the time integration of equation (23) is not necessary so one may construct a homotopy directly for (23).

We assume that the initial approximations $u_{k,0}$, $k = 1, 2$ have the following expansions

$$u_{k,0}\left(t, x\right) = \alpha_{k,0}\left(x\right) + \alpha_{k,1}\left(x\right) t + \alpha_{k,2}\left(x\right) t^2 + \dots, \tag{30}$$

where $\alpha_{k,n}\left(x\right)$, $k = 1, 2$; $n = 0, 1, 2, \dots$ are unknown functions to be determined later on.

In addition, we assume that the solution components v_k of system (28)-(29) can be expressed as a power series in p, as follows

$$v_k\left(t, x\right) = v_{k,0}\left(t, x\right) + p v_{k,1}\left(t, x\right) + p^2 v_{k,2}\left(t, x\right) + \dots, \tag{31}$$

where $v_{k,n}\left(t, x\right)$, $k = 1, 2$; $n = 0, 1, 2, \dots$ are unknown functions to be determined by the following iterative scheme.

Substituting (30) and (31) into system (28)-(29) then equating coefficients of like powers of p, we get the following set of equations

p^0:

$$v_{k,0}(t,x) = u_{k,0}(t,x),\ k = 1,2,$$

p^1:

$$\begin{cases} v_{1,1} = g(x) - u_{1,0} - \int\limits_0^t \phi(u_{1,0}; u_{2,0})\, dt, \\ v_{2,1} = -\psi(u_{1,0}; u_{2,0}), \end{cases}$$

p^i:

$$\begin{cases} v_{1,i} = -\int\limits_0^t \phi(v_{1,0}, \dots, v_{1,i-1}; v_{2,0}, \dots, v_{2,i-1})\, dt, \\ v_{2,i} = v_{2,i-1} - \psi(v_{1,0}, \dots, v_{1,i-1}; v_{2,0}, \dots, v_{2,i-1}),\ i = 2,3,\dots \end{cases}$$

Now, if we set

$$v_{1,1}(t,x) = 0, \tag{32}$$
$$v_{2,1}(t,x) = 0, \tag{33}$$

then all $v_{k,i}(t,x) = 0$, for $k = 1,2$; $i = 2,3,\dots$

To determine the coefficients $\alpha_{k,i}(x)$, $k = 1,2$; $i = 2,3,\dots$ we substitute (30) into system (32)-(33) then solve this system for these coefficients. Finally, using equation (31) we obtain the exact solution components u_k, $k = 1,2$ as series

$$u_k(t,x) = u_{k,0}(t,x) = \sum_{n=0}^{\infty} \alpha_{k,n}(x)\, t^n. \tag{34}$$

The solutions series obtained from NHPM may have limited regions of convergence, even if we take a large number of terms. Therefore, we apply the Laplace-Padé post-treatment to NHPM' truncated series to increase the convergence region.

In the next section, we will apply this modified method to solve a nonlinear index-one PDAE and a linear index-three PDAE with variable coefficients.

6 Test Problems

In this section, we will demonstrate the effectiveness and accuracy of the modified method described in the previous section through two examples of PDAEs. The first example is a nonlinear index-one PDAE whereas the second one is a linear index-three PDAE with variable coefficients.

6.1 Nonlinear index-one system with an integral term

Consider the following nonlinear index-one PDAE which arises as a similarity reduction of Navier-Stokes equations [68]

$$u_{1t} = u_{1xx} - u_2 u_{1x} + u_1^2 - 2\int_0^1 u_1^2 dx, \tag{35}$$
$$0 = u_{2x} - u_1, \tag{36}$$

where $0 < x < 1$ and $t > 0$.

System (35)-(36) is subject to the following initial condition

$$u_1(0,x) = \cos \pi x,\quad 0 \le x \le 1, \tag{37}$$

and boundary conditions

$$u_{1x}(t,0) = u_{1x}(t,1) = \; u_2(t,0) = u_2(t,1) = 0, \quad t \geq 0. \tag{38}$$

The exact solution of problem (35)-(38) is

$$u_1(t,x) = e^{-\pi^2 t}\cos \pi x, \quad u_2(t,x) = (1/\pi)\, e^{-\pi^2 t}\sin \pi x, \quad 0 \leq x \leq 1, \quad t \geq 0. \tag{39}$$

Since one time differentiation of the algebraic equation (36) determines u_{2t} in terms of u and its space derivatives, then PDAE (35)-(36) has time differentiation index-one. Note that no initial condition is prescribed for the variable u_2 as this is determined by the PDAE.

In order to simplify the exposition of the technique discussed in section 4 to solve PDAE (35)-(36), we first integrate equation (35) with respect to time and use the initial condition (37) to get

$$u_1(t,x) - \cos \pi x - \int_0^t \left(u_{1xx} - u_2 u_{1x} + u_1^2 - 2\int_0^1 u_1^2 dx \right) dt = 0. \tag{40}$$

In view of system (28)-(29), the homotopy for (40)-(36) can be constructed as

$$(1-p)(v_1 - u_{1,0}) + p\Bigg(v_1 - \cos \pi x - \tag{41}$$

$$\int_0^t \left(v_{1xx} - v_2 v_{1x} + v_1^2 - 2\int_0^1 v_1^2 dx \right) dt \Bigg) = 0,$$

$$(1-p)(v_2 - u_{2,0}) + p\,(v_{2x} - v_1) = 0, \tag{42}$$

where $p \in [0,1]$ is an embedding parameter.

We assume that the initial approximations $u_{k,0}$ for the solution components u_k, $k = 1,2$ have the form

$$u_{k,0}(t,x) = \alpha_{k,0}(x) + \alpha_{k,1}(x)\,t + \alpha_{k,2}(x)\,t^2 + \ldots, \tag{43}$$

where $\alpha_{k,n}(x)$, $k = 1,2$; $n = 0,1,2,\ldots$ are unknown functions to be determined later on.

We also assume that the solution components v_k of system (41)-(42) can be written as power series in p, as follows

$$v_k(t,x) = v_{k,0}(t,x) + p v_{k,1}(t,x) + p^2 v_{k,2}(t,x) + \ldots, \tag{44}$$

where $v_{k,n}(t,x)$, $k = 1,2$; $n = 0,1,2,\ldots$ are unknown functions to be determined by the following iterative scheme.

Substituting (43) and (44) into system (41)-(42) then equating coefficients of like powers of p, yield the following set of equations

p^0:

$$v_{k,0}(t,x) = u_{k,0}(t,x), \; k = 1,2, \tag{45}$$

p^1:

$$\begin{aligned} v_{1,1}(t,x) &= -\sum_{n=0}^{\infty}\alpha_{1,n}(x)t^n + \cos\pi x + \\ &\int_0^t \sum_{n=0}^{\infty}\alpha''_{1,n}(x)t^n dt - \int_0^t \left(\sum_{n=0}^{\infty}\alpha_{2,n}(x)t^n\right)\times \\ &\left(\sum_{n=0}^{\infty}\alpha'_{1,n}(x)t^n\right)dt + \int_0^t \left(\sum_{n=0}^{\infty}\alpha_{1,n}(x)t^n\right)^2 dt \\ &-2\int_0^t\int_0^1 \left(\sum_{n=0}^{\infty}\alpha_{1,n}(x)t^n dx\right)^2 dt, \end{aligned} \tag{46}$$

$$v_{2,1}(t,x) = \sum_{n=0}^{\infty}\left(-\alpha'_{2,n}(x)+\alpha_{1,n}(x)\right)t^n, \tag{47}$$

where ($'$) denotes the ordinary derivative with respect to x,
p^i:

$$v_{1,i}(t,x) = \int_0^t \left(v_{1,i-1xx} + \sum_{k=0}^{i-1}\gamma_{k,i}(t,x)\right)dt, \tag{48}$$

$$v_{2,i}(t,x) = \int_0^x v_{1,i-1}(t,x)\,dx, \tag{49}$$

where

$$\begin{aligned} \gamma_{k,i}(t,x) &= v_{1,k}v_{1,i-1-k} - v_{2,k}v_{1,i-1-kx} \\ &-2\int_0^1 v_{1,k}v_{1,i-1-k}dx,\ i\geq 2. \end{aligned}$$

Now, if we set

$$v_{1,1}(t,x) = 0, \tag{50}$$

$$v_{2,1}(t,x) = 0, \tag{51}$$

then $v_{1,i}(t,x) = v_{2,i}(t,x) = 0$ for all $i = 2,3,4\ldots$ follows from (48)-(49).

Equating the coefficients of powers of t to zero in (50) then solving the resulting equation for $\alpha_{2,n}(x)$ and using the boundary conditions (38), we have

$$\alpha_{2,n}(x) = \int_0^x \alpha_{1,n}(x)\,dx,\ n = 0,1,2,\ldots \tag{52}$$

Now equation (46) can be written as a series

$$v_{1,1}(t,x) = (-\alpha_{1,0}(x) + \cos\pi x)$$
$$+\sum_{n=1}^{\infty}\left(-\alpha_{1,n}(x) + (1/n)\,\alpha''_{1,n-1}(x) + (1/n)\sum_{k=0}^{n-1}\beta_{k,n}(x)\right)t^n,$$

where

$$\beta_{k,n}(x) = \alpha_{1,k}(x)\,\alpha_{1,n-1-k}(x) - \alpha'_{1,n-1-k}(x)\int_0^x \alpha_{1,k}(x)\,dx$$
$$-2\int_0^1 \alpha_{1,k}(x)\,\alpha_{1,n-1-k}(x)\,dx.$$

Equating all coefficients of powers of t to zero in (50), yields $\alpha_{1,0}(x) = \cos\pi x$ and the recursive formula for $\alpha_{1,n}(x)$

$$\alpha_{1,n}(x) = (1/n)\,\alpha''_{1,n-1}(x) + (1/n)\sum_{k=0}^{n-1}\beta_{k,n}(x),\; n = 1,2,3\ldots \tag{53}$$

From equation (53), we get $\alpha_{1,1}(x) = -\pi^2\cos\pi x$ and $\alpha_{1,2}(x) = \left(\pi^4/2\right)\cos\pi x$.

From equation (52), we obtain $\alpha_{2,0}(x) = (1/\pi)\sin\pi x$, $\alpha_{2,1}(x) = -\pi\sin\pi x$ and $\alpha_{2,2}(x) = \left(\pi^3/2\right)\sin\pi x$.

Using (43) and the coefficients recently computed, we obtain

$$u_1(t,x) = u_{1,0}(t,x) = \left(1 - \pi^2 t + \frac{1}{2}\left(-\pi^2 t\right)^2\right)\cos\pi x, \tag{54}$$

and

$$u_2(t,x) = u_{2,0}(t,x) = \left(1 - \pi^2 t + \frac{1}{2}\left(-\pi^2 t\right)^2\right)(1/\pi)\sin\pi x. \tag{55}$$

In a similar manner, the coefficients $\alpha_{1,n}(x)$ and $\alpha_{2,n}(x)$ for $n \geq 3$ can be obtained from (53) and (52) respectively.

The solutions series obtained from the NHPM may have limited regions of convergence, even if we take a large number of terms. Accuracy can be increased by applying the Laplace-Padé post-treatment. First t-Laplace transform is applied to (54) and (55). Then, s is substituted by $1/t$ and the t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expression to obtain the approximate solution.

Applying Laplace transforms to (54) and (55) yields

$$\mathcal{L}\left[u_1(t,x)\right] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)\cos\pi x, \tag{56}$$

and

$$\mathcal{L}\left[u_2(t,x)\right] = \left(\frac{1}{s} - \frac{\pi^2}{s^2} + \frac{\pi^4}{s^3}\right)(1/\pi)\sin\pi x. \tag{57}$$

For the sake of simplicity we let $s = 1/t$, then

$$\mathcal{L}\left[u_1(t,x)\right] = \left(t - \pi^2 t^2 + \pi^4 t^3\right)\cos\pi x, \tag{58}$$

and

$$\mathcal{L}\left[u_2(t,x)\right] = \left(t - \pi^2 t^2 + \pi^4 t^3\right)(1/\pi)\sin\pi x. \tag{59}$$

All of the $[L/M]$ t-Padé approximants of (58) and (59) with $L \geq 1$ and $M \geq 1$ and $L + M \leq 3$ yield

$$[L/M]_{u_1} = \left(\frac{t}{1 + \pi^2 t}\right)\cos\pi x, \tag{60}$$

and

$$[L/M]_{u_2} = \left(\frac{t}{1+\pi^2 t}\right)(1/\pi)\sin \pi x. \tag{61}$$

Now since $t = 1/s$, we obtain $[L/M]_{u_1}$ and $[L/M]_{u_2}$ in terms of s as follows

$$[L/M]_{u_1} = \left(\pi^2 + s\right)^{-1} \cos \pi x, \tag{62}$$

and

$$[L/M]_{u_2} = \left(\pi^2 + s\right)^{-1} (1/\pi)\sin \pi x. \tag{63}$$

Finally, applying the inverse LT to the Padé approximants (62) and (63), we obtain the modified approximate solution which is in this case the exact solution (39).

Note here that one can take more terms in series (54) and (55) then apply the same procedure described above to find the exact solution (39).

6.2 Index-three system with variable coefficients

Consider the following index-three PDAE system

$$\begin{aligned} u_{1tt} &= u_{1xx} + u_3 \sin \pi x, && (64)\\ u_{2tt} &= u_{2xx} + u_3 \cos \pi x, && (65)\\ 0 &= u_1 \sin \pi x + u_2 \cos \pi x - e^{-t}, && (66) \end{aligned}$$

where $t > 0$ and $0 < x < 1$.

System (64)-(66) is subject to the following initial conditions

$$\begin{aligned} u_1(0,x) &= \sin \pi x, \;\; u_{1t}(0,x) = -\sin \pi x, && (67)\\ u_2(0,x) &= \cos \pi x, \;\; u_{2t}(0,x) = -\cos \pi x, \;\; 0 \le x \le 1, && (68) \end{aligned}$$

and the boundary conditions

$$u_1(t,0) = u_1(t,1) = 0, \; u_2(t,0) = -u_2(t,1) = e^{-t}, \;\; t \ge 0. \tag{69}$$

The exact solution of problem (64)-(69) is

$$\begin{aligned} u_1(t,x) = e^{-t}\sin \pi x, \;\; u_2(t,x) = e^{-t}\cos \pi x, \;\; u_3(t,x) = \left(1+\pi^2\right)e^{-t},\\ 0 \le x \le 1, \quad t \ge 0. \end{aligned} \tag{70}$$

Since three time differentiations of equation (66) determine u_{3t} in terms of the solution u and its space derivatives, then PDAE (64)-(66) is index-three. Therefore, this PDAE is difficult to solve numerically. Moreover, no initial condition is prescribed for the variable u_3 as this is determined by the PDAE.

In order to simplify the exposition of the technique discussed in section 4 to solve (64)-(69), we first integrate equations (64) and (65) twice with respect to time and use the initial conditions (67)-(68) to get

$$u_1(t,x) - \sin \pi x + t\sin \pi x - \int_0^t \int_0^t u_{1xx} + u_3 \sin \pi x \; dtdt = 0, \tag{71}$$

$$u_2(t,x) - \cos \pi x + t\cos \pi x - \int_0^t \int_0^t u_{2xx} + u_3 \cos \pi x \; dtdt = 0. \tag{72}$$

In view of system (28)-(29), the homotopy for set of equations (71), (72) and (66) can be constructed as follows

$$\begin{aligned}
&(1-p)(v_1 - u_{1,0}) + p\Bigg(v_1 - \sin\pi x + t\sin\pi x \\
&- \int_0^t \int_0^t v_{1xx} + v_3 \sin\pi x \; dtdt \Bigg) = 0, \\
&(1-p)(v_2 - u_{2,0}) + p\Bigg(v_2 - \cos\pi x + t\cos\pi x \\
&- \int_0^t \int_0^t v_{2xx} + v_3 \cos\pi x \; dtdt \Bigg) = 0, \\
&(1-p)(v_3 - u_{3,0}) + p\Bigg(v_1(t,x)\sin\pi x + \\
&v_2(t,x)\cos\pi x - e^{-t} \Bigg) = 0,
\end{aligned} \tag{73}$$

where $p \in [0,1]$ is an embedding parameter.

We assume that the initial approximation $u_{k,0}$ for the exact solution components u_k, $k = 1,2,3$ has the form

$$u_{k,0}(t,x) = \alpha_{k,0}(x) + \alpha_{k,1}(x)\,t + \alpha_{k,2}(x)\,t^2 + \ldots, \tag{74}$$

where $\alpha_{k,n}(x)$, $k = 1,2,3$; $n = 0,1,2,\ldots$ are unknown functions to be determined later on.

We assume also that the solution components v_k of system (73) can be written as power series in p, as follows

$$v_k(t,x) = v_{k,0}(t,x) + p v_{k,1}(t,x) + p^2 v_{k,2}(t,x) + \ldots, \tag{75}$$

where $v_{k,n}(t,x)$, $k = 1,2,3$; $n = 0,1,2,\ldots$ are unknown functions to be determined by the following iterative scheme. Substituting (74) and (75) into system (73) then equating coefficients of like powers of p yield the following set of equations

p^0:

$$v_{k,0}(t,x) = u_{k,0}(t,x),\; k = 1,2,3,$$

p^1:

$$\begin{aligned}
v_{1,1}(t,x) \quad = \quad & -\sum_{n=0}^{\infty} \alpha_{1,n}(x)\,t^n + \sin\pi x - t\sin\pi x \\
& + \int_0^t \int_0^t \sum_{n=0}^{\infty} \alpha''_{1,n}(x)\,t^n dtdt \\
& + \sin\pi x \int_0^t \int_0^t \sum_{n=0}^{\infty} \alpha_{3,n}(x)\,t^n dtdt,
\end{aligned} \tag{76}$$

$$\begin{aligned} v_{2,1}(t,x) &= -\sum_{n=0}^{\infty}\alpha_{2,n}(x)\,t^n + \cos\pi x - t\cos\pi x \\ &+ \int_0^t\int_0^t \sum_{n=0}^{\infty}\alpha''_{2,n}(x)\,t^n dtdt \\ &+ \cos\pi x \int_0^t\int_0^t \sum_{n=0}^{\infty}\alpha_{3,n}(x)\,t^n dtdt, \end{aligned} \tag{77}$$

$$v_{3,1}(t,x) = e^{-t} - v_{1,0}(t,x)\sin\pi x - v_{2,0}(t,x)\cos\pi x, \tag{78}$$

where $(')$ denotes the ordinary derivative with respect to x,
p^i:

$$\begin{aligned} v_{1,i}(t,x) &= \int_0^t\int_0^t v_{1,i-1xx} + v_{3,i-1}\sin\pi x dtdt, \\ v_{2,i}(t,x) &= \int_0^t\int_0^t v_{2,i-1xx} + v_{3,i-1}\cos\pi x dtdt, \\ v_{3,i}(t,x) &= v_{3,i-1} - v_{1,i-1}\sin\pi x - v_{2,i-1}\cos\pi x,\ i=2,3,\ldots \end{aligned} \tag{79}$$

Now, if we set

$$\begin{aligned} v_{1,1}(t,x) &= 0, \\ v_{1,2}(t,x) &= 0, \\ v_{1,3}(t,x) &= 0, \end{aligned} \tag{80}$$

then $v_{1,i}(t,x) = v_{2,i}(t,x) = v_{3,i}(t,x) = 0$ for all $i = 2,3,\ldots$ follows from (79).

Equations (76)-(78) can be rewritten as series

$$\begin{aligned} v_{1,1}(t,x) &= (-\alpha_{1,0}(x) + \sin\pi x) - (\alpha_{1,1}(x) + \sin\pi x)\,t + \\ & \sum_{n=2}^{\infty}\left(\frac{\alpha''_{1,n-2}(x) + \alpha_{3,n-2}(x)\sin\pi x}{(n-1)\,n} - \alpha_{1,n}(x)\right)t^n, \end{aligned}$$

$$\begin{aligned} v_{2,1}(t,x) &= (-\alpha_{2,0}(x) + \cos\pi x) - (\alpha_{2,1}(x) + \cos\pi x)\,t + \\ & \sum_{n=2}^{\infty}\left(\frac{\alpha''_{2,n-2}(x) + \alpha_{3,n-2}(x)\cos\pi x}{(n-1)\,n} - \alpha_{2,n}(x)\right)t^n, \end{aligned}$$

$$v_{3,1}(t,x) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{n!} - \alpha_{1,n}(x)\sin\pi x - \alpha_{2,n}(x)\cos\pi x\right)t^n.$$

Equating all coefficients of powers of t to zero in (80) yields

$$\begin{aligned} \alpha_{1,0}(x) &= \sin\pi x, \quad \alpha_{1,1}(x) = -\sin\pi x, \\ \alpha_{2,0}(x) &= \cos\pi x, \quad \alpha_{2,1}(x) = -\cos\pi x, \end{aligned}$$

and the nonsingular algebraic linear system for the unknown functions $\alpha_{1,n}$, $\alpha_{2,n}$ and $\alpha_{3,n-2}$

$$\begin{aligned} \alpha_{1,n}(x) - \frac{\alpha_{3,n-2}(x)\sin\pi x}{(n-1)n} &= \frac{\alpha''_{1,n-2}(x)}{(n-1)n}, \\ \alpha_{2,n}(x) - \frac{\alpha_{3,n-2}(x)\cos\pi x}{(n-1)n} &= \frac{\alpha''_{2,n-2}(x)}{(n-1)n}, \\ \alpha_{1,n}(x)\sin\pi x + \alpha_{2,n}(x)\cos\pi x &= \frac{(-1)^n}{n!}, \text{ for } n = 2, 3, \ldots \end{aligned} \tag{81}$$

Solving system (81) exactly, we have the recursions

$$\begin{aligned} \alpha_{1,n}(x) &= \frac{(-1)^n}{n!}\sin\pi x + \frac{\delta_n(x)\cos\pi x}{(n-1)n}, \\ \alpha_{2,n}(x) &= \frac{(-1)^n}{n!}\cos\pi x - \frac{\delta_n(x)\sin\pi x}{(n-1)n}, \\ \alpha_{3,n-2}(x) &= \frac{(-1)^n}{(n-2)!} - \alpha''_{1,n-2}(x)\sin\pi x - \alpha''_{2,n-2}(x)\cos\pi x, \end{aligned} \tag{82}$$

where $\delta_n(x) = \alpha''_{1,n-2}(x)\cos\pi x - \alpha''_{2,n-2}(x)\sin\pi x$, for $n = 2, 3, \ldots$

For $n = 2, 3, 4$, we have $\delta_n(x) = 0$ and hence

$$\alpha_{1,2}(x) = \frac{1}{2}\sin\pi x,\ \alpha_{2,2}(x) = \frac{1}{2}\cos\pi x, \alpha_{3,0}(x) = 1 + \pi^2,$$

$$\alpha_{1,3}(x) = -\frac{1}{6}\sin\pi x,\ \alpha_{2,3}(x) = -\frac{1}{6}\cos\pi x,\ \alpha_{3,1}(x) = -\left(1+\pi^2\right),$$

$$\alpha_{1,4}(x) = \frac{1}{24}\sin\pi x,\ \alpha_{2,4}(x) = \frac{1}{24}\cos\pi x,\ \alpha_{3,2}(x) = \frac{1}{2}\left(1+\pi^2\right).$$

Using (74) and the coefficients recently computed, we obtain

$$u_1(t,x) = u_{1,0}(t,x) = \left(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right)\sin\pi x, \tag{83}$$

$$u_2(t,x) = u_{2,0}(t,x) = \left(1 - t + \frac{1}{2}t^2 - \frac{1}{3!}t^3 + \frac{1}{4!}t^4\right)\cos\pi x, \tag{84}$$

and

$$u_3(t,x) = u_{3,0}(t,x) = \left(1+\pi^2\right)\left(1 - t + \frac{1}{2}t^2\right). \tag{85}$$

In a similar manner, the coefficients $\alpha_{1,n}(x)$, $\alpha_{2,n}(x)$ and $\alpha_{3,n-2}(x)$ for $n \geq 5$ can be obtained from (82).

The solutions series obtained from the NHPM may have limited regions of convergence, even if we take a large number of terms. Therefore, we apply the t-Padé approximation technique to these series to increase the convergence region. First t-Laplace transform is applied to (83), (84) and (85). Then, s is substituted by $1/t$ and the t-Padé approximant is applied to the transformed series. Finally, t is substituted by $1/s$ and the inverse Laplace s-transform is applied to the resulting expression obtain the modified approximate solution.

Applying Laplace transform to (83), (84) and (85) yields

$$\mathcal{L}\left[u_1(t,x)\right] = \left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)\sin\pi x, \tag{86}$$

$$\mathcal{L}\left[u_2(t,x)\right] = \left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right)\cos\pi x, \tag{87}$$

and

$$\mathcal{L}\left[u_3\left(t,x\right)\right] = \left(1+\pi^2\right)\left(\frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3}\right). \tag{88}$$

For the sake of simplicity we let $s = 1/t$, then

$$\mathcal{L}\left[u_1\left(t,x\right)\right] = \left(t - t^2 + t^3\right)\sin \pi x, \tag{89}$$

$$\mathcal{L}\left[u_2\left(t,x\right)\right] = \left(t - t^2 + t^3\right)\cos \pi x, \tag{90}$$

and

$$\mathcal{L}\left[u_3\left(t,x\right)\right] = \left(1+\pi^2\right)\left(t - t^2 + t^3\right). \tag{91}$$

All of the $[L/M]$ t-Padé approximants of (89), (90) and (91) with $L \geq 1$ and $M \geq 1$ and $L+M \leq 3$ yield

$$[L/M]_{u_1} = \left(\frac{t}{1+t}\right)\sin \pi x, \tag{92}$$

$$[L/M]_{u_2} = \left(\frac{t}{1+t}\right)\cos \pi x, \tag{93}$$

and

$$[L/M]_{u_3} = \left(1+\pi^2\right)\left(\frac{t}{1+t}\right). \tag{94}$$

Now since $t = 1/s$, we obtain $[L/M]_{u_1}$, $[L/M]_{u_2}$ and $[L/M]_{u_3}$ in terms of s as follows

$$[L/M]_{u_1} = (1+s)^{-1}\sin \pi x, \tag{95}$$

$$[L/M]_{u_2} = (1+s)^{-1}\cos \pi x, \tag{96}$$

and

$$[L/M]_{u_3} = \left(1+\pi^2\right)(1+s)^{-1}. \tag{97}$$

Finally, applying the inverse Laplace transform to the Padé approximants (95), (96) and (97), we obtain the modified approximate solution which is in this case the exact solution (70).

7 Discussion

In this paper we presented new homotopy perturbation method (NHPM) as a useful analytical tool to solve partial differential-algebraic equations (PDAEs). Two PDAE problems were solved determining the exact solutions; the first one was a nonlinear index-one problem and the second one was a variable coefficients linear index-three problem. The method has successfully handled the index-three PDAE without the need for a preprocessing step of index-reduction. For each of the two problems solved here, the NHPM transformed the PDAE into an easily solvable linear algebraic system for the coefficient functions of the power series solution. To improve the NHPM solution, a Laplace-Padé (LP) post-treatment is applied to the truncated series leading to the exact solution. Additionally, the solution procedure does not involve unnecessary computation like that related to noise terms [39]. This greatly reduces the volume of computation and improves the efficiency of the method. It should be noticed that the high complexity of these problems was effectively handled by Laplace-Padé new homotopy perturbation method (LPNHPM) due to the malleability of NHPM and resummation capability of Laplace-Padé. What is more, there is not any standard analytical or numerical methods to solve higher-index PDAEs, converting the LPNHPM method into an attractive tool to solve such problems.

On one hand, semi-analytic methods like HPM, HAM, VIM among others, require an initial approximation for the sought solutions and the computation of one or several adjustment parameters. If the initial approximation is properly chosen the results can be highly accurate, nonetheless, no general methods are available to choose such initial approximation. This issue motivates the use of adjustment parameters obtained by minimizing the least-squares error with respect to the numerical solution.

On the other hand, NHPM or LPNHPM methods do not require any trial equation as requisite for the starting the method. What is more, NHPM obtains its coefficients using an easy computable straightforward procedure that can be implemented into programs like Maple or Mathematica. Finally, if the solution of the PDAE is not expressible in terms of known functions then the LP resummation will provide a larger domain of convergence.

8 Conclusion

In this paper, a hybrid method which combines a new homotopy perturbation method (NHPM), Laplace transform (LT), Padé approximant (PA) is introduced for solving PDAEs analytically. Solutions of PDAEs are first obtained in convergent series forms using a new homotopy perturbation method (NHPM). To improve the solutions obtained from NHPM's truncated series, a post-treatment combining Laplace transform and Padé approximant is proposed. This modified Laplace-Padé new homotopy perturbation method (LPNHPM) greatly improves NHPM's truncated series solutions in convergence rate, and often leads to the exact solution. Two examples of PDAEs are given to demonstrate this result. Additionally, the LPNHPM does not require any index-reduction to solve higher-index PDAEs and the solutions do not involve noise terms which may reduce the efficiency of the method. The LPNHPM proposed in this paper is expected to be further employed to solve other PDAE systems.

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Competing Interests

Authors have declared that no competing interests exist.

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