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On the Sum of Reciprocals of Mersenne Primes

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Abstract

The sum of reciprocals of Mersenne primes converges to $0.51645417894078856533\cdots$, which is an example of a probably infinite subset of primes whose sum of reciprocals is finite and can be computed accurately. This value is larger than $\sum_{\{q \in \mathbb{P} | p \geq 3\}} \dfrac{1}{q-1}$, where $\mathbb{P}$ is the set of perfect powers of prime numbers.

Keywords

Mersenne Primes, the Sum of Reciprocals, Unresolved Problem

1. Introduction

Since monk Marine Mersenne studied the primality of $M_n = 2^n - 1$ in 1644, Mersenne primes, *i.e.*, $M_n = 2^n - 1$ (M_n: prime), have been developed by numerous researchers, such as Euler, Lucas, Pervouchine, Cole, and Powers, and in recent years, by GIMPS (Great Internet Mersenne Prime Search).

If M_n is prime, then n is also prime, because if $n = ab$, ($a, b \geq 2$), then $M_n = 11\cdots1$ (ab digits in binary) can be divided by $1\cdots1$ (a digits in binary). However, the converse is not true, for example, $M_{11} = 2047 = 23 \times 89$.

In addition, it is well known that all even perfect numbers (odd perfect numbers are unknown) are generated by $2^{n-1} M_n$, if and only if M_n is prime.

The current Mersenne prime numbers are denoted by M_n, for $n = 2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, \cdots$. The most recent Mersenne prime number is $M_{74207281}$ (22338618 digits), which was developed in January 2016.

2. Bounds for the Sum of Reciprocals of Mersenne Primes

We begin by defining the notation. We define

$$S \equiv \{2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, \cdots\}$$

i_k: k-th number in S.

Theorem 1.

$$\sum_{k=1}^{N} \frac{1}{M_{ik}} < \sum_{M_n:\text{primes}} \frac{1}{M_n} < \sum_{k=1}^{N} \frac{1}{M_{ik}} + 2^{2-i_{N+1}}, \ \forall i_N \in S$$

Proof.

$$\sum_{M_n:\text{primes}} \frac{1}{M_n} = \sum_{i \in S} \frac{1}{M_i} = \sum_{k=1}^{N} \frac{1}{M_{ik}} + \sum_{k=N+1}^{\infty} \frac{1}{M_{ik}} < \sum_{k=1}^{N} \frac{1}{M_{ik}} + \sum_{k=N+1}^{\infty} \frac{1}{2^{i_k} - 1}$$

$$< \sum_{k=1}^{N} \frac{1}{M_{ik}} + \sum_{k=N+1}^{\infty} \frac{1}{2^{i_k-1}} = \sum_{k=1}^{N} \frac{1}{M_{ik}} + 2^{2-i_{N+1}}$$

We can effectively calculate $\sum_{M_n:\text{primes}} \dfrac{1}{M_n}$, as $2^{2-i_{N+1}}$ rapidly converges to 0.

For example, if we consider $N = 8$, we obtain

$$0.51645417894078856489\cdots < \sum_{M_n:\text{primes}} \frac{1}{M_n} < 0.51645417894078856663\cdots,$$

which provides the value of $\sum_{M_n:\text{primes}} \dfrac{1}{M_n}$ up to 17 decimal digits. If we con-

sider $N = 12$, we can precisely calculate the sum of reciprocals of Mersenne primes up to 156 decimal digits, which is given by

0.51645417894078856533048734297152285881596855341541970144193106527356870144021272349915488329366621537403243211083656957541914047092486831748603728529464163 4⋯

3. Comments

According to the Goldbach-Euler theorem [1],

$$\sum_{q \in \mathbb{P}} \frac{1}{q-1} = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \cdots = 1,$$

where $\mathbb{P} \equiv \left\{ p^i \mid p \text{ is a prime number} \geq 2 \text{ and } i \geq 2 \right\}$ is the set of perfect powers of prime numbers.

Theorem 2. *The sum of reciprocals of Mersenne prime numbers is larger than that of $q-1$ where $q \in \mathbb{P} \mid p \geq 3$, namely,*

$$\sum_{M_n:\text{primes}} \frac{1}{M_n} > \sum_{\{q \in \mathbb{P} \mid p \geq 3\}} \frac{1}{q-1}$$

Proof. It holds that

$$\frac{1}{M_2} + \frac{1}{M_3} + \frac{1}{M_5} \approx 0.50844854\cdots < \sum_{M_n:\text{primes}} \frac{1}{M_n}.$$

Considering that $M_n : \text{primes} \in 2^i - 1 \ (i \geq 2)$, it follows from Goldbach-Euler theorem that

$$\sum_{M_n:\text{primes}} \frac{1}{M_n} + \sum_{\{q \in \mathbb{P} \mid p \geq 3\}} \frac{1}{q-1} < \sum_{i \geq 3} \frac{1}{2^i - 1} + \sum_{\{q \in \mathbb{P} \mid p \geq 3\}} \frac{1}{q-1} = 1.$$
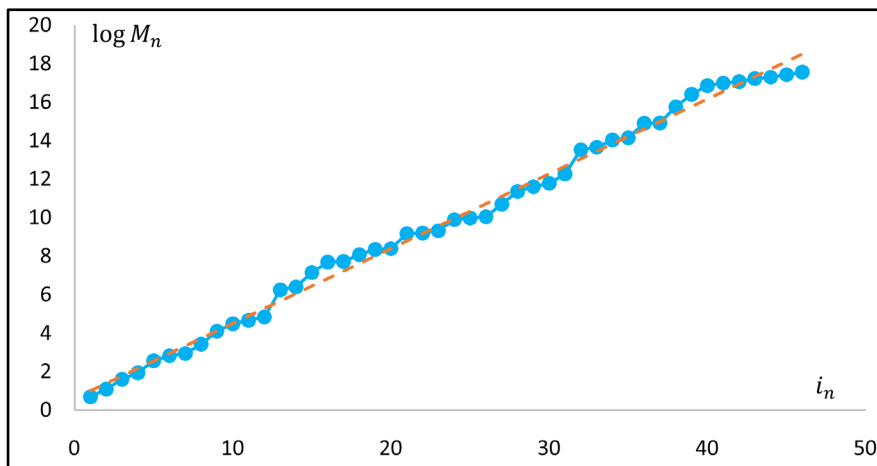
Hence,

Figure 1. Relationship between i_n and $\log M_n$.

$$\sum_{\{q\in\mathbb{P}\,|\,p\geq 3\}} \frac{1}{q-1} < \frac{1}{2} < \sum_{M_n:\text{primes}} \frac{1}{M_n},$$

since $\dfrac{1}{2} < \displaystyle\sum_{M_n:\text{primes}} \dfrac{1}{M_n}$.

We should note that the sum of reciprocals of prime numbers appears to converge numerically; however, it is infinite, which is proved in, e.g., Hardy and Wright [2]. Therefore, Mersenne primes are considerably sparse subsequences of prime numbers.

In the case of twin primes, the value of the sum of reciprocals of twin primes is shown to be bounded above by Brun [3] and is estimated as

$$\left(\frac{1}{3}+\frac{1}{5}\right)+\left(\frac{1}{5}+\frac{1}{7}\right)+\left(\frac{1}{11}+\frac{1}{13}\right)+\left(\frac{1}{17}+\frac{1}{19}\right)+\cdots \approx 1.9021605823 \pm 8\times 10^{-10},$$

which is known as Brun's constant (however, it is an *estimation*). Even though the problem that whether twin primes are infinite is still unsolved, Zhang [4] presented an important result, which states that a constant $< 7\times 10^7$ exists between two successive primes that are infinite. If we can lower the upper bound by 4, the twin prime conjecture will be solved. The Polymath8 project has reduced the upper bound by four digits.

In addition, the problem that whether Mersenne primes are infinite is still unresolved. **Figure 1** shows the relationship between i_n and $\log M_n$.

Denote $m(x)$ the number of Mersenne primes that do not exceed x. Then, as $m(x)=i_n$ for $x=M_n$, it seems from **Figure 1** that

$$m(x) \propto \log x + c,$$

numerically, where c is a constant. In other words, the number of Mersenne primes tends to increase by a constant per digit.

References

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[3] Brun, V. (1919) La série $\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19}+\frac{1}{29}+\frac{1}{31}+\frac{1}{41}+\frac{1}{43}+\frac{1}{59}+\frac{1}{61}+\cdots$ les dénominateurs sont nombres premiers jumeaux est convergente où finie. *Bulletin des Sciences Mathématiques*, **43**, 124-128. (In French)

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