British Journal of Mathematics & Computer Science
11(1): 1-16, 2015, Article no.BJMCS.19199
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

Indirect RBF for High-Order Integro-Differential Equations

Jafar Biazar[1*] and Mohammad Ali Asadi[1]

[1] *Department of Applied Mathematics, Faculty of Mathematical Science, University of Guilan, P.O. Box 41635-19141, P.C. 41938-33697, Rasht, Iran.*

Article Information

DOI: 10.9734/BJMCS/2015/19199
Editor(s):
(1) Narayan Thapa, Department of Mathematics and Computer Science, Minot State University, Minot, USA.
Reviewers:
(1) Anonymous, Turkey.
(2) Mohammed H. AL-Smadi, Dept. of Applied Science, Al-Balqa Applied University, Jordan.
(3) Anonymous, University of Cartagena, Colombia.
Complete Peer review History: http://sciencedomain.org/review-history/10462

Original Research Article

Received: 29 May 2015
Accepted: 01 July 2015
Published: 09 August 2015

Abstract

Two different approaches are applied to solve high-order integro-differential equations (IDEs), based on Radial Basis Functions (RBF). The first approach, which is called the direct approach (DRBF) is based on differentiation, and considers the solution as a finite linear combination of RBFs. While the second, the indirect approach (IRBF), is based on integration and considers the highest order derivative of the solution as a finite linear combination of RBFs. The results of this study indicate that for low-order IDEs, both approaches are enough accurate, but for high-order IDEs, the IRBF solutions are more accurate than those of direct RBF.
Illustrative examples are included to demonstrate the validity and applicability of the presented technique.

Keywords: High order integro-differential equations (IDEs); Direct Radial Basis Functions (DRBF); Indirect Radial Basis Functions (IRBF); Legendre-Gauss-Lobatto quadrature; Iterated integrals.
2010 Mathematics Subject Classification: 65R20, 45J05.

**Corresponding author E-mail: biazar@guilan.ac.ir*

1 Introduction

Integral and integro-differential equations arise from various applications, like physics, engineering, biology, medicine, economics, potential theory and many others (See [1, 2, 3] and references therein). Since many of these equations can not be solved explicitly, it is often necessary to use numerical techniques. In recent years, a lot of attention has been devoted to the study of high-order integro-differential equations (IDEs) such as the Bernoulli matrix method [4], the collocation method [5], Chebyshev polynomials [6, 7], Legendre polynomials [8, 9, 10], Homotopy Analysis Method [11], Homotopy Perturbation Method [12], Adomian Decomposition Method [13], Variational iteration method [14, 15, 16], Taylor polynomials [17, 18, 19], Compact Finite Difference Method [20], reproducing kernel method [21, 22, 23, 24, 25, 26], radial basis functions (RBF) [27] and RBF Networks (RBFN) [28].

In recent years, meshless methods as a class of numerical methods are used for solving functional equations. Meshless methods just use a scattered set of collocation points, regardless any relationship information between the collocation points. This property is the main advantage of these techniques over the mesh dependent methods such as finite difference methods and finite element methods. Since 1990, radial basis function method [29] are used as a well-known family of meshless method to approximate the solutions of various types of linear and nonlinear functional equations such as Partial Differential Equations (PDEs), Ordinary Differential Equations (ODEs), Integral Equations (IEs), and Integro-Differential Equation (IDEs) [29, 30, 31, 32, 33, 34, 35].

In this paper, a modification of radial basis functions (RBF) is applied for numerical solution of high-order integro-differential equations which is based on integration approach (IRBF) and first introduced and used by Mai-Duy and Tran-Cong [36, 37, 38]. So far IRBF method has only been applied to solve first-order IDEs [27]. This paper will focus on generalizing this method for high-order IDEs. Iterated integrals appeared in this approach are converted to one integral by using a formula of iterated integrals. A comparison is made between this approach and the differentiation approach of RBF (DRBF). We investigate these two approaches on the high order integro-differential equations and results demonstrate the good accuracy and efficiency of the presented technique for higher order IDEs than low order IDEs.

The paper is organized as follows. In Section 2, the radial basis functions are introduced. Section 3, reviews the Legendre-Gauss-Lobatto integration process. Section 4, as the main part, presents the solution of high-order integro-differential equations by direct and indirect process of radial basis functions. Numerical illustrative examples are included in Section 5. A conclusion is drawn in the Section 6.

2 Radial Basis Functions

Let's define the main features of the method.

2.1 Definition of Radial Basis Functions

Radial basis functions usually approximate a function as the following [39]

$$s(x) = \sum_{i=0}^{N} \lambda_i \phi(\|x - x_i\|), \quad x \in \mathbb{R}.$$

Where $\phi : [0, \infty) \to \mathbb{R}$ is a fixed univariate function, the coefficients $(\lambda_i)_{i=0}^{N}$ are real numbers, $(x_i)_{i=0}^{N}$, are finite number of distinct points (centers) in $\mathbb{R}$ and $\|\cdot\|$, denotes the Euclidean norm.

2.2 Radial Basis Functions Interpolation

The radial basis functions approximation of a real function, say $u(x)$, is given by [39]

$$u(x) \approx u_N(x) = \sum_{i=0}^{N} \lambda_i \phi\left(\|x - x_i\|\right) = \sum_{i=0}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda},$$

where

$$\phi_i(x) = \phi(\|x - x_i\|),$$
$$\mathbf{\Phi}^T(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_N(x)],$$
$$\mathbf{\Lambda} = [\lambda_0, \lambda_1, ...\lambda_N]^T,$$

and distinct points $(x_i)_{i=0}^N$ are in $\mathbb{R}$. Consider N distinct support points $(x_j, u(x_j))$, $j = 0, 1, 2, ..., N$. One can find λ_is by solving the following linear system

$$\mathbf{A\Lambda} = \mathbf{u}.$$

Where

$$\mathbf{A} = [\phi(\|x_j - x_i\|)]_{i,j=0}^N,$$

$\mathbf{\Lambda} = [\lambda_0, \lambda_1, ...\lambda_N]^T$, and $\mathbf{u} = [u_0, u_1, ...u_N]^T$. Some well-known RBFs are listed in Table 1, where the Euclidian distance r is real and non-negative, and c is a positive scalar, called shape parameter.

Table 1: Some well-known RBFs

Name of the function	Definition
Gaussian	$\phi(r) = e^{-(cr)^2}$
Inverse Quadric	$\phi(r) = \frac{1}{r^2+c^2}$
Hardy Multiquadric	$\phi(r) = \sqrt{r^2 + c^2}$
Inverse Multiquadric	$\phi(r) = \frac{1}{\sqrt{r^2+c^2}}$
Cubic	$\phi(r) = r^3$
Thin plate splines	$\phi(r) = r^2 \log(r)$
Hyperbolic secant	$\phi(r) = \text{sech}(cr)$

3 Legendre-Gauss-Lobatto Integration Nodes and Weights

Let L_N be the well-known Legendre polynomial of order N, on the interval $[-1, 1]$. Then the Legendre-Gauss-Lobatto nodes are

$$(1 - x_j^2)L'_N(x_j) = 0,$$
$$-1 = x_0 < x_1 < \ldots < x_N = 1, \tag{3.1}$$

where $x_m, 1 \leq m \leq N - 1$ are the zeros of L'_N, where L'_N is the derivative of L_N with respect to $x \in [-1, 1]$. No explicit formula for the nodes (3.1) is known, and so they are computed numerically using sub-routines [40, 41, 42]. Now we approximate the integral of f on $[-1, 1]$ as

$$\int_{-1}^{1} f(x)\, dx = \sum_{i=0}^{N} w_j f(x_j), \tag{3.2}$$

where x_j in Eq. (3.1) are Legendre-Gauss-Lobatto nodes and w_j are the weights given in [43]

$$w_j = \frac{2}{N(N+1)} \frac{1}{(L_N(x_j))^2}, \quad j = 0, 1, \ldots, N.$$

Note that the integration in Eq.(3.2) is exact whenever $f(x)$ is a polynomial of degree $\leq 2N+1$.

4 Application of RBF Method

In this paper Radial Basis Functions are used to approximate solution of high order integro-differential equations of the second kind in the following general form

$$y^{(m)}(x) + p(x) y^{(r)}(x) = f(x) + \int_a^b K(x,t) y^{(s)}(t)\, dt, \quad a \leq x \leq b, \tag{4.1}$$

$$y(a) = \alpha_0,\ y'(a) = \alpha_1,\ \ldots,\ y^{(m-1)}(a) = \alpha_{m-1} \tag{4.2}$$

where $y(x)$ is an unknown real function defined on $[a,b]$, $r, s \leq m$, and $p(x)$ and $f(x)$ are analytic known functions. Moreover the kernel $K(x,t)$ is defined on the interval $a \leq x, t \leq b$.

4.1 Direct Radial Basis Functions

In the direct method, the unknown function of Eq. (4.1) approximates by a closed form of radial basis functions, and its derivatives of any order, e.g. n-th order, can then be calculated by differentiating such a closed form.

Let's approximate the function $y(x)$ in terms of radial basis functions, $\phi_i(x)$, as follows

$$y(x) \approx y_N(x) = \sum_{i=0}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda}, \tag{4.3}$$

where

$$\phi_i(x) = \phi(\|x - x_i\|),$$

x_i, $i = 0, 1, ..., N$ are shifted nodes of Legendre-Gauss-Lobatto quadrature,

$$\mathbf{\Phi}^T(x) = [\phi_0(x), \phi_1(x), \ldots, \phi_N(x)]$$

and

$$\mathbf{\Lambda} = [\lambda_0, \lambda_1, \ldots, \lambda_N]^T$$

is an unknown vector.

By m times differentiating from Eq. (4.3), we obtain

$$y^{(m)}(x) \approx y_N^{(m)}(x) = \mathcal{D}^m \mathbf{\Phi}^T(x)\mathbf{\Lambda} \tag{4.4}$$

where

$$\mathcal{D}^m \mathbf{\Phi}^T(x) = [\phi_0^{(m)}(x), \phi_1^{(m)}(x), \ldots, \phi_N^{(m)}(x)]$$

Obviously $y^{(r)}$ and $y^{(s)}$ are calculated.

Substituting $y^{(m)}$, $y^{(r)}$, and $y^{(s)}$ in Eq. (4.1), leads to

$$\mathcal{D}^m \mathbf{\Phi}^T(x)\mathbf{\Lambda} + p(x)\mathcal{D}^r \mathbf{\Phi}^T(x)\mathbf{\Lambda} = f(x) + \int_a^b K(x,t)\mathcal{D}^s \mathbf{\Phi}^T(t)\mathbf{\Lambda} dt, \quad a \leq x \leq b,$$

or

$$\left(\mathcal{D}^m \mathbf{\Phi}^T(x) + p(x)\,\mathcal{D}^r \mathbf{\Phi}^T(x) - \int_a^b K(x,t)\mathcal{D}^s \mathbf{\Phi}^T(t)dt\right)\mathbf{\Lambda} = f(x).$$

For obtaining λ_i, $i = 0, 1, \ldots, N$, by collocating at the points $x = x_j$ for $j = 0, 1, \ldots, N$, we have

$$\left(\mathcal{D}^m \mathbf{\Phi}^T(x_j) + p(x_j)\,\mathcal{D}^r \mathbf{\Phi}^T(x_j) - \int_a^b K(x_j,t)\mathcal{D}^s \mathbf{\Phi}^T(t)dt\right)\mathbf{\Lambda} = f(x_j), \tag{4.5}$$

where x_j, $j = 0, 1, ..., N$ are shifted zeros of the Legendre-Gauss-Lobatto integration nodes.
By using the Legendre-Gauss-Lobatto integration formula, we can approximate the integral in Eq. (4.5) and hence these equations can be written as follows:

$$\left(\mathcal{D}^m \mathbf{\Phi}^T(x_j) + p(x_j)\,\mathcal{D}^r \mathbf{\Phi}^T(x_j) - \sum_{i=0}^{N} w_i\, K(x_j,t_i)\,\mathcal{D}^s \mathbf{\Phi}^T(t_i)\right)\mathbf{\Lambda} = f(x_j), \tag{4.6}$$

for $j = 0, 1, \ldots, N$. Where $t_i \in [a, b]$ and w_i, for $i = 0, \ldots, N$ are Legendre-Gauss-Lobatto integration nodes and weights, respectively. Eq. (4.6) generates a system of linear equations for the unknowns $\mathbf{\Lambda}$.
Many researchers substitute initial conditions

$$\begin{aligned}
\mathbf{\Phi}^T(a) &= \alpha_0,\\
\mathcal{D}\mathbf{\Phi}^T(a) &= \alpha_1,\\
&\vdots\\
\mathcal{D}^{m-1}\mathbf{\Phi}^T(a) &= \alpha_{m-1},
\end{aligned}$$

for the same number of equation in the foregoing linear system. There is not any criterion for such substitution and seems it is the author's option. This freedom usually reduces the accuracy.

4.2 Indirect Radial Basis Functions

In order to apply indirect radial basis functions approach, let's approximate the highest order derivative in terms of radial basis functions $\phi_i(x)$ as follows

$$y^{(m)}(x) \approx \hat{y}_N = \sum_{i=0}^{N} \lambda_i \phi_i(x) = \mathbf{\Phi}^T(x)\mathbf{\Lambda} \tag{4.7}$$

Successive integrating the obtained expression yields expressions for lower order derivatives and finally for the original function itself. For example

$$\begin{aligned}
\int_a^x y^{(m)}(t)dt &= y^{(m-1)}(x) - y^{(m-1)}(a)\\
&\approx \sum_{i=0}^{N} \lambda_i \int_a^x \phi_i(t)\,dt = \mathcal{I}\mathbf{\Phi}^T(x)\mathbf{\Lambda}
\end{aligned}$$

or

$$y^{(m-1)}(x) \approx \mathcal{I}\mathbf{\Phi}^T(x)\mathbf{\Lambda} + y^{(m-1)}(a),$$

where

$$\mathcal{I}\mathbf{\Phi}^T(x) = \left[\int_a^x \phi_0(t)\,dt, \int_a^x \phi_1(t)\,dt, \ldots, \int_a^x \phi_N(t)\,dt\right].$$

Similarly

$$y^{(m-2)}(x) \approx \sum_{i=0}^{N} \lambda_i \int_a^x \int_a^{t_2} \phi_i(t)\, dt\, dt_2 + y^{(m-1)}(a)x + y^{(m-2)}(a)$$
$$= \mathcal{I}^2 \mathbf{\Phi}^T(x)\mathbf{\Lambda} + y^{(m-1)}(a)x + y^{(m-2)}(a),$$

and finally

$$y \approx \mathcal{I}^m \mathbf{\Phi}^T(x)\mathbf{\Lambda} + y^{(m-1)}(a)\frac{x^{m-1}}{(m-1)!} + y^{(m-2)}(a)\frac{x^{m-2}}{(m-2)!} + \cdots + y'(a)x + y(a) \tag{4.8}$$

where the i-th component of $\mathcal{I}^m \mathbf{\Phi}^T(x)$ is

$$\int_a^x \int_a^{t_m} \cdots \int_a^{t_3} \int_a^{t_2} \phi_i(t)\, dt\, dt_2 \ldots\, dt_{m-1}\, dt_m$$

and $y(a), y'(a), \ldots, y^{(m-1)}(a)$ are known from initial conditions.
Also we have

$$y^{(r)}(x) \approx \mathcal{I}^{m-r} \mathbf{\Phi}^T(x)\mathbf{\Lambda} + y^{(m-1)}(a)\frac{x^{m-r-1}}{(m-r-1)!} + y^{(m-2)}(a)\frac{x^{m-r-2}}{(m-r-2)!} + \cdots + y^{(r-1)}(a)x + y^{(r)}(a) \tag{4.9}$$

and

$$y^{(s)}(x) \approx \mathcal{I}^{m-s} \mathbf{\Phi}^T(x)\mathbf{\Lambda} + y^{(m-1)}(a)\frac{x^{m-s-1}}{(m-s-1)!} + y^{(m-2)}(a)\frac{x^{m-s-2}}{(m-s-2)!} + \cdots + y^{(s-1)}(a)x + y^{(s)}(a) \tag{4.10}$$

Substituting from Eqs. (4.7), (4.9), and (4.10) in Eq. (4.1), leads to

$$\mathbf{\Phi}^T(x)\mathbf{\Lambda} + p(x)\mathcal{I}^{m-r}\mathbf{\Phi}^T(x)\mathbf{\Lambda} + p(x)\left(y^{(m-1)}(a)\frac{x^{m-r-1}}{(m-r-1)!} + y^{(m-2)}(a)\frac{x^{m-r-2}}{(m-r-2)!} + \cdots + y^{(r-1)}(a)x + y^{(r)}(a)\right)$$
$$= f(x) + \int_a^b K(x,t)\, \mathcal{I}^{m-s}\mathbf{\Phi}^T(t)\mathbf{\Lambda} dt + \int_a^b K(x,t)\left(y^{(m-1)}(a)\frac{t^{m-s-1}}{(m-s-1)!} + y^{(m-2)}(a)\frac{t^{m-s-2}}{(m-s-2)!} + \cdots + y^{(s-1)}(a)t + y^{(s)}(a)\right)dt,$$

or

$$\left(\mathbf{\Phi}^T(x) + p(x)\,\mathcal{I}^{m-r}\mathbf{\Phi}^T(x) - \int_a^b K(x,t)\mathcal{I}^{m-s}\mathbf{\Phi}^T(t)dt\right)\mathbf{\Lambda} = g(x), \tag{4.11}$$

in which

$$\begin{aligned} g(x) &= f(x) \\ &- p(x)\left(y^{(m-1)}(a)\frac{x^{m-r-1}}{(m-r-1)!} + y^{(m-2)}(a)\frac{x^{m-r-2}}{(m-r-2)!}\right. \\ &\left. + \cdots + y^{(r-1)}(a)x + y^{(r)}(a)\right) \\ &+ \int_a^b K(x,t)\left(y^{(m-1)}(a)\frac{t^{m-s-1}}{(m-s-1)!} + y^{(m-2)}(a)\frac{t^{m-s-2}}{(m-s-2)!}\right. \\ &\left. + \cdots + y^{(s-1)}(a)t + y^{(s)}(a)\right)dt, \end{aligned}$$

Fortunately, integrals on left hand side of (4.11), can be reduced to one dimensional integrals by using the formula of iterated integrals [44], for example

$$\int_a^x \int_a^{t_m} \cdots \int_a^{t_3} \int_a^{t_2} \phi_i(t)\, dt\, dt_2 \ldots dt_{m-1}\, dt_m = \frac{(x-a)^m}{(m-1)!} \int_0^1 t^{m-1} \phi_i\left(x-(x-a)\,t\right)\, dt \quad (4.12)$$

Now numerical quadratures can be applied.

For calculating λ_i, $i = 0, 1, \ldots, N$, we apply collocation process similar to direct approach by using Legendre-Gauss-Lobatto nodes as collocation points. Integrals appeared in process, determined by Legendre-Gauss-Lobatto quadrature. Finally approximate solution of Eq. (4.1), is given by (4.8).

In the integration process all initial conditions are considered in the RBF expansion of solution and extra equations for initial conditions do not required. Furthermore, in the differentiation process, initial conditions substituted for the same number of equations and this substitution usually reduces the accuracy. Indeed it is expected that through the indirect process, the approximating functions are much smoother and therefore have higher approximation power.

The convergence of radial basis function interpolation has been discussed by [39, 45] and other researchers [46, 47, 48].

5 Numerical Examples

In this section, some examples are provided to illustrate the efficiency of this approach. For the sake of comparing purposes, we use the norm two of errors.

5.1 Example

Consider the following second order Fredholm IDE

$$y''(x) = -e^x + \frac{x}{2} + \int_0^1 xty(t), \quad y(0) = 0,\ y'(0) = -1, \quad 0 \le x \le 1. \quad (5.1)$$

The exact solution is $y(x) = 1 - e^x$.
Errors of the numerical solutions for $N = 5, 10, 15$ and GA-RBF, MQ-RBF, and IMQ-RBF are shown in Table 2 and Fig. 1,2, and 3.

5.2 Example

Consider the following third order Fredholm IDE

$$y'''(x) = sin(x) - x - \int_0^{\frac{\pi}{2}} xty'(t)dt, , \quad 0 \le x \le \frac{\pi}{2}. \quad (5.2)$$

Table 2: Errors for Example 5.1

	GA		MQ		IMQ	
N	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	6.1815e-01	3.6710e-06	1.4967e+00	8.3951e-06	2.5269e+00	4.1628e-05
10	3.5582e-04	4.9120e-09	3.3468e-02	2.8375e-09	1.7722e-01	1.5954e-08
15	1.3948e-05	3.1498e-08	3.2377e-04	4.3236e-09	1.9503e-03	7.6985e-10

with the initial conditions $y(0) = 1$, $y'(0) = 0$, and $y''(0) = -1$. The exact solution is $y(x) = \cos(x)$. Errors of the numerical solutions for $N = 5, 10, 15$ and same RBFs as Example 5.1 are shown in Table 3 and Fig. 4,5, and 6.

Table 3: Errors for Example 5.2

	GA		MQ		IMQ	
N	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	5.0037e-01	9.2822e-06	5.9669e-01	1.8861e-05	6.8239e-01	1.0677e-04
10	5.6828e-02	1.1815e-10	2.2210e+00	1.0834e-08	1.1974e+00	6.2370e-08
15	5.9581e-02	3.4710e-10	8.0248e-01	8.3737e-11	1.3336e+00	4.4014e-10

5.3 Example

Now consider the following fourth-order Fredholm IDE

$$y^{(4)} = -1 + sin(x) + \int_0^{\frac{\pi}{2}} ty(t)dt, \quad 0 \leq x \leq \frac{\pi}{2}. \tag{5.3}$$

with the initial conditions $y(0) = y''(0) = 0$, $y'(0) = 1$, and $y'''(0) = -1$. The exact solution is $y(x) = \sin(x)$.
Errors of the numerical solutions for $N = 5, 10, 15$ and different RBFs are shown in Table 4 and Fig. 7,8, and 9.

Table 4: Errors for Example 5.3

	GA		MQ		IMQ	
N	DRBF	IRBF	DRBF	IRBF	DRBF	IRBF
5	9.2288e-01	3.4373e-06	2.0403e+01	1.2329e-05	1.0582e+01	5.3253e-05
10	3.4521e-02	6.0687e-12	3.4740e-01	5.1621e-09	6.3145e+01	2.6527e-08
15	7.3037e-03	4.2092e-10	3.3438e-01	5.6452e-12	9.6491e-02	1.4849e-11

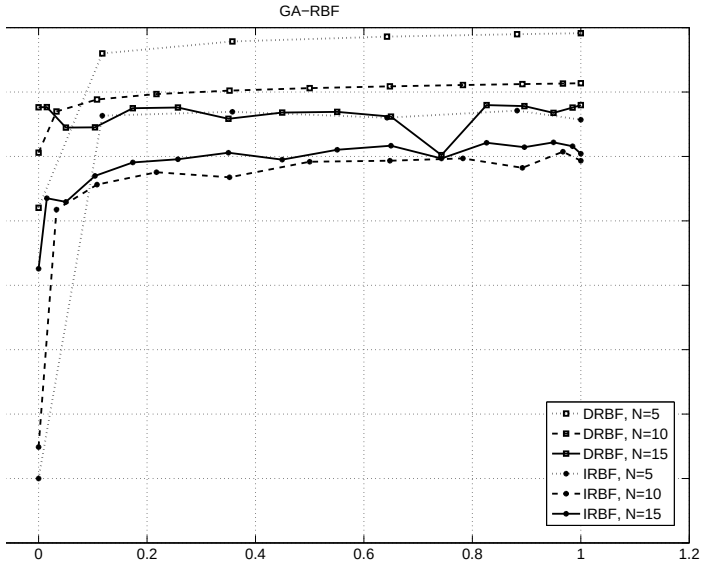

Figure 1: $\log|y_{Exact} - y_{Approx}|$ for Example 5.1 by GA-RBF

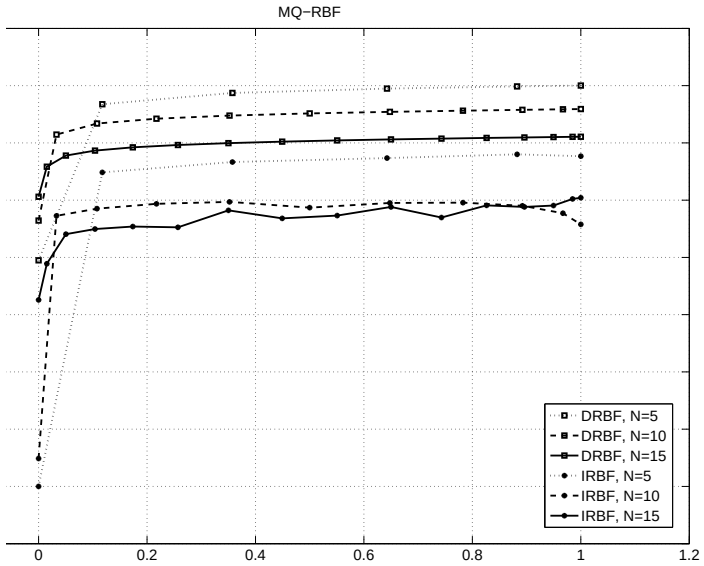

Figure 2: $\log|y_{Exact} - y_{Approx}|$ for Example 5.1 by MQ-RBF

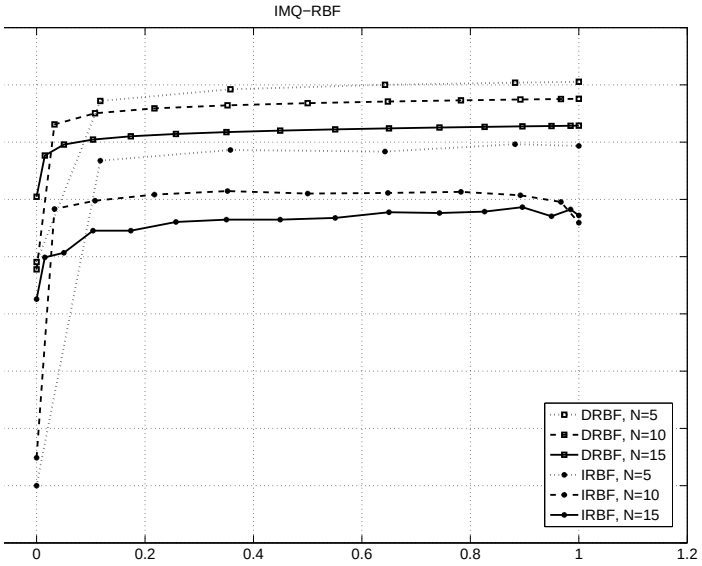

Figure 3: $\log|y_{Exact} - y_{Approx}|$ for Example 5.1 by IMQ-RBF

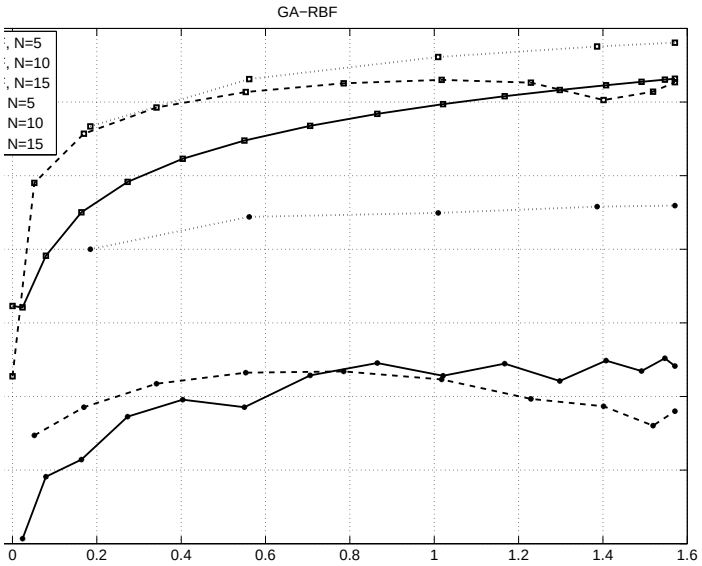

Figure 4: $\log|y_{Exact} - y_{Approx}|$ for Example 5.2 by GA-RBF

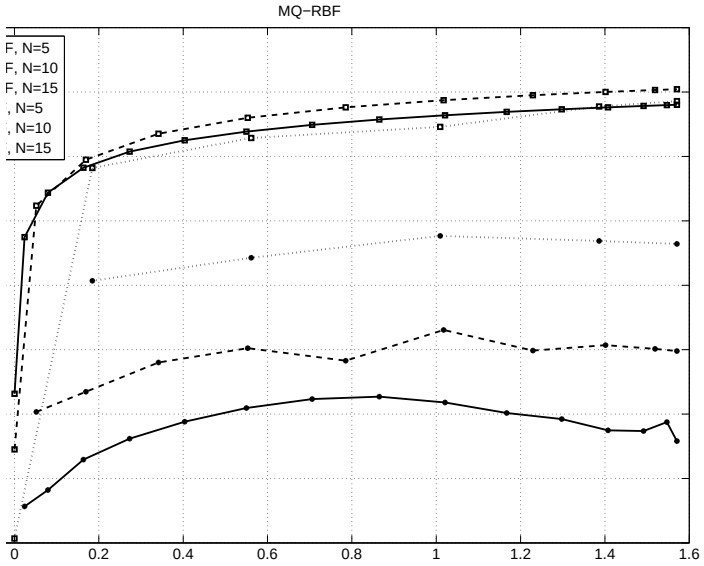

Figure 5: $\log|y_{Exact} - y_{Approx}|$ for Example 5.2 by MQ-RBF

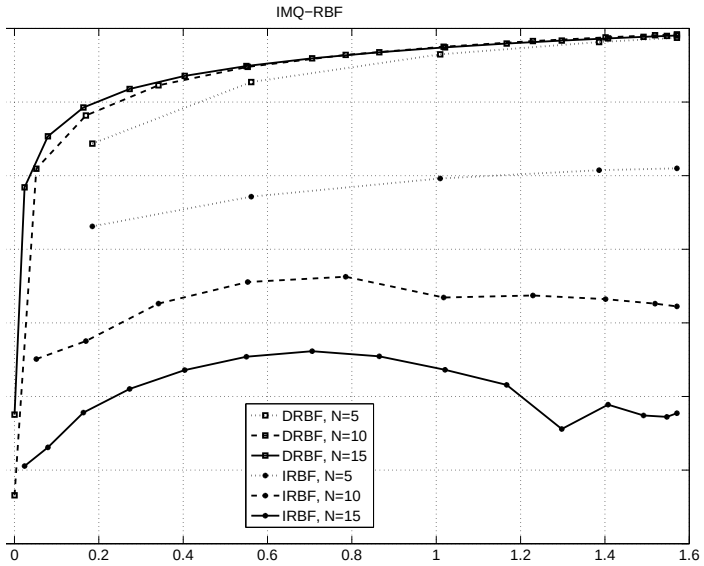

Figure 6: $\log|y_{Exact} - y_{Approx}|$ for Example 5.2 by IMQ-RBF

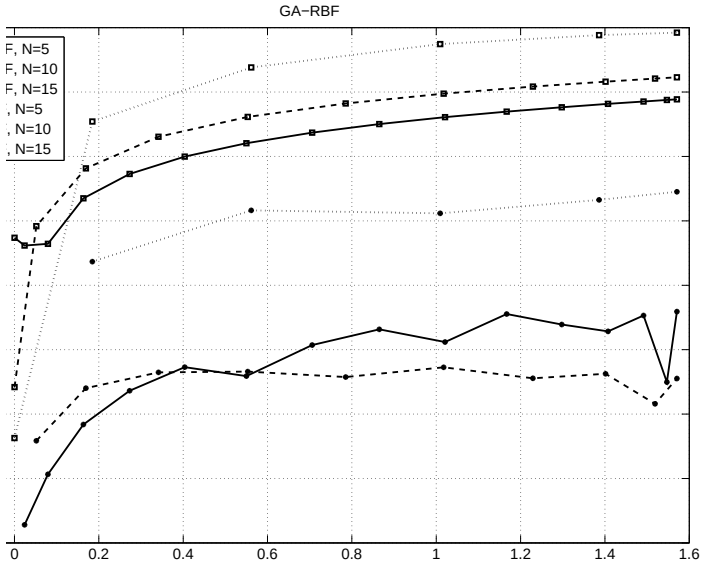

Figure 7: $\log|y_{Exact} - y_{Approx}|$ for Example 5.3 by GA-RBF

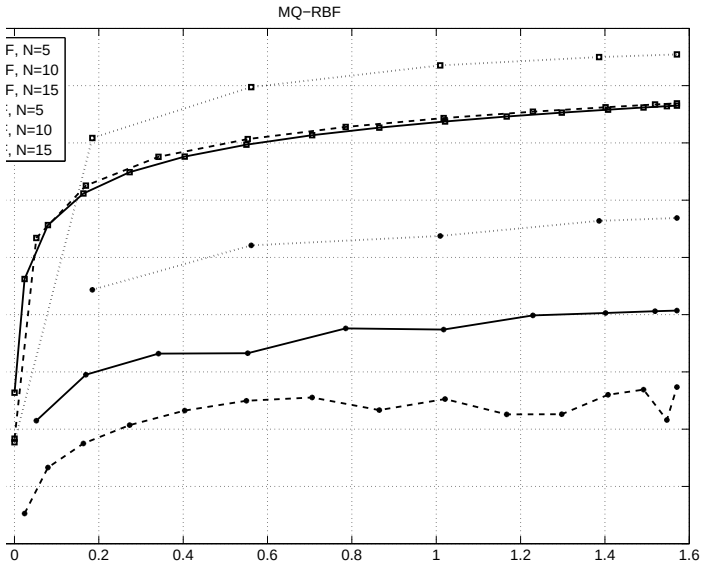

Figure 8: $\log|y_{Exact} - y_{Approx}|$ for Example 5.3 by MQ-RBF

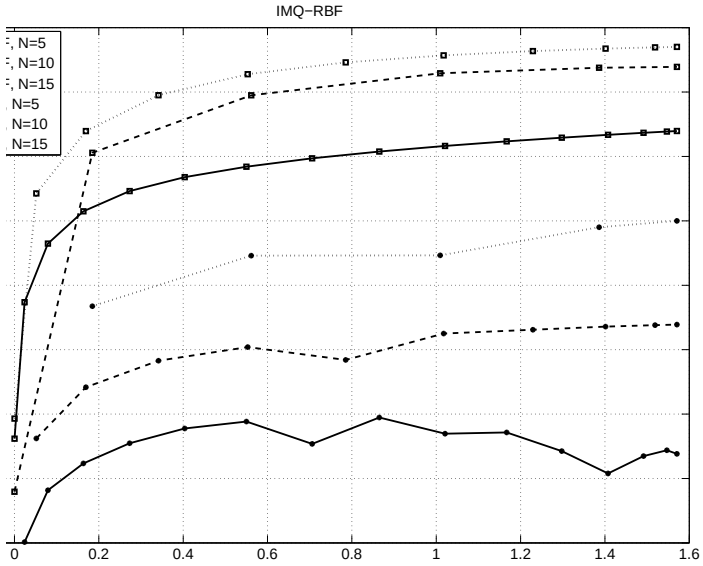

Figure 9: $\log|y_{Exact} - y_{Approx}|$ for Example 5.3 by IMQ-RBF

6 Conclusion

A modified radial basis function approach was used for solving high-order integro-differential equations. Some numerical examples are presented to demonstrate that the method is very effective and useful for finding approximate solutions of high order integro-differential equations. A comparison with direct process, for numerical solution of the high-order integro-differential equations, shows that this technique is accurate enough to be known as a powerful device. Figures and tables show that the IRBF solutions are more accurate than RBF solutions. Especially, as the order of integro-differential equations is increased, the difference of the graphs of error for IRBF and DRBF methods increased.

Competing Interests

The authors declare that no competing interests exist.

References

[1] Wazwaz AM. Linear and nonlinear integral equations. Methods and Applications; 2011.

[2] Delves LM, Mohamed JL. Computational methods for integral equations. Cambridge University Press; 1985.

[3] Kress R. Linear integral equations.

[4] Bhrawy AH, Tohidi E, Soleymani F. A new bernoulli matrix method for solving high-order linear and nonlinear fredholm integro-differential equations with piecewise intervals. Appl. Math. Comput. 2012;219(2):482-497.

[5] Yüzbaşı I Ş , Niyazi ahn, Ahmet Yildirim. A collocation approach for solving high-order linear fredholmvolterra integro-differential equations. Math. Comput. Model. 2012;55(34):547-563.

[6] Akyüz-Daşcoğlu A, Sezer M. Chebyshev polynomial solutions of systems of higher-order linear fredholmvolterra integro-differential equations. J. Franklin Inst. 2005;342(6):688-701.

[7] El-gendi SE. Chebyshev solution of differential, integral and integro-differential equations. Comput. J.. 1969;12(3):282-287.

[8] YalçinbaşS, Sezer M, Sorkun HH. Legendre polynomial solutions of high-order linear fredholm integro-differential equations. Appl. Math. Comput. 2009;210(2):334-349.

[9] Sohrabi S. An efficient spectral method for high-order nonlinear integro-differential equations. U.P.B. Sci. Bull., Series A. 2012;74(3):75-88.

[10] Saadatmandi A, Dehghan M. Numerical solution of the higher-order linear fredholm integro-differential-difference equation with variable coefficients. Comput. Math. Appl. 2010;59(8):2996-3004.

[11] Jaradat HM, Awawdeh F, Alsayyed O. Series solution to the high-order integro-differential equations. Analele Uni. Din Oradea. Fasc. Matematica, Tom. 2009;16:247-257.

[12] Aminikhah H, Biazar J. Exact solution for high-order integro-differential equations by nhpm. Int. J. Nonlinear Sci. 2009;7(4):496-500.

[13] Wazwaz AM. A reliable algorithm for solving boundary value problems for higher-order integro-differential equations. Appl. Math. Comput. 2001;118(23):327-342.

[14] Shang X, Han D. Application of the variational iteration method for solving th-order integro-differential equations. J. Comput. Appl. Math. 2010;234(5):1442-1447.

[15] Noor MA, Mohyud-Din ST. A reliable approach for higher-order integro-differential equations. Appl. Appl. Math. 2008;3(2):188-199.

[16] Wang SQ, He JH. Variational iteration method for solving integro-differential equations. Phys. Lett. A. 2007;367(3):188-191.

[17] Yalçinbaş S, Sezer M. The approximate solution of high-order linear volterrafredholm integro-differential equations in terms of taylor polynomials. Appl. Math. Comput. 2000;112(23):291-308.

[18] Nas S, YalçinbaşS, Sezer M. A taylor polynomial approach for solving high order linear fredholm integro-differential equations. Int. J. Math. Educ. Sci. Technol. 2000;31(2):213-225.

[19] Maleknejad K, Mahmoudi Y. Taylor polynomial solution of high-order nonlinear volterrafredholm integro-differential equations.Appl. Math. Comput. 2003;145(23):641-653.

[20] Zhao J. Compact finite difference methods for high order integro-differential equations. Appl. Math. Comput. 2013;221:66-78.

[21] Al-Smadi M, Abu-Arqub O, Shawagfeh N. Approximate solution of bvps for 4th-order ides by using rkhs method. Appl. Math. Sci. 2012;6(50):2453-2464.

[22] Abu-Arqub O, Al-Smadi M, Momani S. Application of reproducing kernel method for solving nonlinear fredholm-volterra integrodifferential equations. Abstr. Appl. Anal. 2012;16:2012. Article ID: 839836.

[23] Abu-Arqub O, Al-Smadi M, Shawagfeh N. Solving fredholm integrodifferential equations using reproducing kernel hilbert space method. Appl. Math. Comput. 2013;219(17):8938-8948.

[24] Al-Smadi M, Abu-Arqub O, Momani S. A computational method for two-point boundary value problems of fourth-order mixed integrodifferential equations. Math. Probl. Eng. 2013;10:2013. Article ID: 832074.

[25] Komashynska I, AL-Smadi M. Iterative reproducing kernel method for solving second-order integrodifferential equations of fredholm type. J. Appl. Math. 2014;2014.

[26] Abu-Arqub O, Al-Smadi M. Numerical algorithm for solving two-point, second-order periodic boundary value problems for mixed integro-differential equations. Appl. Math. Comput. 2014;243:911-922.

[27] Parand K, Abbasbandy S, Kazem S, Rad JA. A novel application of radial basis functions for solving a model of first-order integro-ordinary differential equation. Commun. Nonlinear Sci. Numer. Simul. 2011;16(11):4250-4258.

[28] Golbabai A, Seifollahi S. Radial basis function networks in the numerical solution of linear integro-differential equations.Appl. Math. Comput. 2007;188(1):427-432.

[29] Hardy RL. Multiquadric equations of topography and other irregular surfaces. J. Geophys. Res. 1971;76(8):1905-1915.

[30] Kansa EJ. Multiquadrics-a scattered data approximation scheme with applications to computational fluid-dynamics-ii solutions to parabolic, hyperbolic and elliptic partial differential equations. Comput. Math. Appl. 1990;19(8-9):147-161.

[31] Platte RB, Driscoll TA. Computing eigenmodes of elliptic operators using radial basis functions. Comput. Math. Appl. 2004;48:561-576.

[32] Larsson E, Fornberg B. A numerical study of some radial basis function based solution methods for elliptic {PDEs}. Comput. Math. Appl. 2003;46(5-6):891–902.

[33] Dehghan M, Shokri A. A numerical method for two-dimensional schrdinger equation using collocation and radial basis functions. Comput. Math. Appl. 2007;54(1):136-146.

[34] Dehghan M, Shokri A. A numerical method for solution of the two-dimensional sine-gordon equation using the radial basis functions. Math. Comput. Simul. 2008;79(3):700-715.

[35] Franke C, Schaback R. Solving partial differential equations by collocation using radial basis functions. Appl. Math. Comput. 1997;93:73-82.

[36] Mai-Duy N, Tran-Cong T. Numerical solution of differential equations using multiquadric radial basis function networks. Neural Networ. 2001;14(2):185-199.

[37] Mai-Duy N, Tran-Cong T. Approximation of function and its derivatives using radial basis function networks. Appl. Math. Model. 2003;27(3):197-220.

[38] Mai-Duy N. Solving high order ordinary differential equations with radial basis function networks. Int. J. Numer. Meth. Eng. 2005;62(6):824-852.

[39] Buhmann MD. Radial basis functions: theory and implementations. Cambridge University Press; 2003.

[40] Canuto C, Hussaini MY, Quarteroni A, Zang TA. Spectral methods in fluid dynamics. Springer series in computational physics. Springer-Verlag; 1988.

[41] Elnagar GN, Kazemi MA. Pseudospectral legendre-based optimal computation of nonlinear constrained variational problems. J. Comput. Appl. Math. 1998;88(2):363-375.

[42] Elnagar GN, Razzaghi M. Short communication: A collocation-type method for linear quadratic optimal control problems. Optim. Control. Appl. Meth. 1997;18(3):227-235.

[43] Shen J, Tang T. High order numerical methods and algorithms. Chinese Science Press; 2005.

[44] Abramowitz M, Stegun IA.Handbook of mathematical functions. Dover Publications, New York; 1972.

[45] Buhmann MD. Multivariable interpolation using radial basis functions. PhD thesis, University of Cambridge; 1989.

[46] Jackson IRH. Radial basis function methods for multivariable approximation. PhD thesis, University of Cambridge; 1988.

[47] Schaback R. Approximation by radial basis functions with finitely many centers. Constr. Approx., 1996;12(3):331-340.

[48] Wendland H. Error estimates for interpolation by compactly supported radial basis functions of minimal degree. J. Approx. Theory. 1998;93(2):258-272.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/10462