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# On the non-linear diophantine equation $379^x + 397^y = z^2$

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**Abstract:** In this article, authors discussed the existence of solution of non-linear diophantine equation $379^x + 397^y = z^2$, where $x, y, z$ are non-negative integers. Results show that the considered non-linear diophantine equation has no non-negative integer solution.

**Keywords:** Prime number, diophantine equation, solution, integers.

**MSC:** 11D61, 11D72, 11D45.

## 1. Introduction

Diophantine equations are those equations which are to be solved in integers. Diophantine equations are very important equations of theory of numbers and have many important applications in algebra, analytical geometry and trigonometry. These equations give us an idea to prove the existence of irrational numbers [1,2]. Acu [3] studied the diophantine equation $2^x + 5^y = z^2$ and proved that $\{x = 3, y = 0, z = 3\}$ and $\{x = 2, y = 1, z = 3\}$ are the solutions of this equation. Kumar *et al.*, [4] considered the non-linear diophantine equations $61^x + 67^y = z^2$ and $67^x + 73^y = z^2$. They showed that these equations have no non-negative integer solution. Kumar *et al.* [5] studied the non-linear diophantine equations $31^x + 41^y = z^2$ and $61^x + 71^y = z^2$ and determined that these equations have no non-negative integer solution. Rabago [6] discussed the open problem given by B. Sroysang. He showed that the diophantine equation $8^x + p^y = z^2$, where $x, y, z$ are positive integers has only three solutions namely $\{x = 1, y = 1, z = 5\}$, $\{x = 2, y = 1, z = 9\}$ and $\{x = 3, y = 1, z = 23\}$ for $p = 17$. The diophantine equations $8^x + 19^y = z^2$ and $8^x + 13^y = z^2$ were studied by Sroysang [7,8]. He proved that these equations have a unique non-negative integer solution namely $\{x = 1, y = 0, z = 3\}$. Sroysang [9] proved that the diophantine equation $31^x + 32^y = z^2$ has no non-negative integer solution.

The main aim of this article is to discuss the existence of solution of non-linear diophantine equation $379^x + 397^y = z^2$, where $x, y, z$ are non-negative integers.

## 2. Preliminary

**Lemma 1.** *The non-linear diophantine equation $379^x + 1 = z^2$, where $x, z$ are non-negative integers, has no solution in non-negative integers.*

**Proof.** Since 379 is an odd prime, so $379^x$ is an odd number for all non-negative integer $x$, which implies $379^x + 1 = z^2$ is an even number for all non-negative integer $x$, so $z$ is an even number. Hence

$$z^2 = 0\,(mod 3) \quad \text{or} \quad z^2 = 1\,(mod 3)\,. \tag{1}$$

Now, $379 = 1\,(mod 3)$, which implies $379^x = 1\,(mod 3)$, for all non-negative integer $x$, so $379^x + 1 = 2\,(mod 3)$, for all non-negative integer $x$. Hence

$$z^2 = 2\,(mod 3)\,. \tag{2}$$

Equation (2) contradicts Equation (1). Hence non-linear diophantine equation $379^x + 1 = z^2$ has no non-negative integer solution. □

**Lemma 2.** *The non-linear Diophantine equation $397^y + 1 = z^2$, where $y, z$ are for all non-negative integers, has no solution in non-negative integers.*

**Proof.** Since 397 is an odd prime, so $397^y$ is an odd number for all non-negative integer $y$, which implies $397^y + 1 = z^2$ is an even number for all non-negative integer $y$, so $z$ is an even number. Hence

$$\Rightarrow z^2 = 0\,(mod 3)\, \text{or} z^2 = 1\,(mod 3)\,. \tag{3}$$

Now, $397 = 1\,(mod3)$, implies $397^y = 1\,(mod3)\,$, for all non-negative integer $y$, so $397^y + 1 = 2\,(mod3)\,$, for all non-negative integer $y$. Hence

$$z^2 = 2\,(mod3)\,. \tag{4}$$

Equation (4) contradicts Equation (3). Hence non-linear Diophantine equation $397^y + 1 = z^2$ has no non-negative integer solution. □

**Theorem 1** (Main theorem)**.** *The non-linear diophantine equation $379^x + 397^y = z^2$, where $x, y, z$ are non-negative integers, has no solution in non-negative integers.*

**Proof.** There are following four cases;

Case 1. If $x = 0$ then the non-linear diophantine equation $379^x + 397^y = z^2$ becomes $1 + 397^y = z^2$, which has no non-negative integer solution by Lemma 2.

Case 2. If $y = 0$ then the non-linear diophantine equation $379^x + 397^y = z^2$ becomes $379^x + 1 = z^2$, which has no non-negative integer solution by Lemma 1.

Case 3. If $x, y$ are positive integers, then $379^x, 397^y$ are odd numbers, implies $379^x + 397^y = z^2$ is an even number, so $z$ is an even number, Hence

$$z^2 = 0\,(mod3)\, orz^2 = 1\,(mod3)\,. \tag{5}$$

Now, $379 = 1\,(mod3)$, implies $379^x = 1\,(mod3)$ and $397^y = 1\,(mod3)$, so $379^x + 397^y = 2\,(mod3)$. Hence

$$z^2 = 2\,(mod3)\,. \tag{6}$$

Equation (6) contradicts Equation (5). Hence non-linear Diophantine equation $379^x + 397^y = z^2$ has no non-negative integer solution.

Case 4. If $x, y = 0$, then $379^x + 397^y = 1 + 1 = 2 = z^2$, which is not possible because $z$ is a non-negative integer. Hence diophantine equation $379^x + 397^y = z^2$ has no non-negative integer solution.

□

## 3. Conclusion

In this article, authors successfully discussed the solution of non-linear diophantine equation $379^x + 397^y = z^2$, where $x, y, z$ are non-negative integers and determined that this non-linear equation has no non-negative integer solution.

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