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The Kumaraswamy-Burr Type III Distribution: Properties and Estimation

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

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Abstract

The Burr Type III distribution attracts special attention in life testing and reliability analysis as it is applied in several areas such as economics and environmetrics among others. A composite distribution of Kumaraswamy and Burr Type III distributions, referred to as Kumaraswamy-Burr Type III distribution, is introduced and studied. It contains some special well-known distributions, which are discussed in lifetime literature, such as the Burr Type III, exponentiated Burr Type III and Kumaraswamy-Burr Type XII, among several others. Some properties of the proposed distribution are studied including explicit expressions for the moments, the density functions of the order statistics, Rényi entropy, quantiles and moment generating function. The method of maximum likelihood is applied under Type II censored samples for estimating the model parameters, reliability and hazard rate functions. For different values of sample sizes, Monte Carlo simulation is performed to investigate the precision of the maximum likelihood estimates.

Keywords: Kumaraswamy distribution; stress-strength; reversed hazard rate function; censored sampling; maximum likelihood method; asymptotic information matrix.

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1 Introduction

[1] Constructed a distribution with two shape parameters on (0, 1) which is known as Kumaraswamy distribution and denoted by Kum (a, b). Its cumulative distribution function (cdf) is defined by

$$H(y) \equiv H(y;a,b) = 1-(1-y^a)^b \;, \quad y \in (0,1), \tag{1}$$

where $a, b > 0$ are shape parameters.

[1] argued that the beta distribution does not faithfully fit hydrological random variables such as daily rainfall, daily stream flow, etc. He developed a more general probability density function (pdf) for double bounded random processes. It has simple explicit formulae for the distribution and quantile functions. This distribution is applicable to many natural phenomena whose outcomes have lower and upper bounds, such as the heights of individuals, scores obtained on a test, atmospheric temperatures and hydrological data. The Kum distribution is considered a better alternative to the beta distribution in hydrology and related areas (see [2,3,4]).

Burr Type III distribution attracts special attention since it includes several families of nonnormal distributions (e.g. gamma distribution) and it includes the characteristics of other distributions such as logistic and exponential distributions. This distribution has been widely applied in various fields such as environmental studies, survival and reliability analysis, forestry, economics, meteorology and water resources among others. It is suitable to fit lifetime data since it has flexible shape and controllable scale parameters. Its distribution is used to study of income, wages and wealth as it is employed in financial literature. Its distribution can be used to simulate random sampling from a normal distribution. [5,6] focused on the distribution in terms of derivation and properties and the estimation through Bayesian and non-Bayesian estimation and also Bayesian prediction (see [7,8]).

[9] used a composite distribution function, F, as a generated function by composing a cdf H with another cdf G. Kumaraswamy-Burr Type III (Kum-BIII(a,b,c,k)) distribution is a composite distribution of Kum distribution with parameters (a, b) and Burr Type III (c, k) (Burr III).

[10] proposed the Kumaraswamy generalized distribution (Kum-G) with the following cdf and pdf respectively

$$F(x) = H[G(x)] = 1-[1-\{G(x)\}^a]^b, \qquad -\infty < x < \infty; \quad a,b > 0, \tag{2}$$

and

$$f(x;a,b) = abg(x)\,(G(x))^{a-1}[1-\{G(x)\}^a]^{b-1}, \qquad -\infty < x < \infty; \quad a,b > 0. \tag{3}$$

The density family in (3) has many more properties better than the class of generalized beta distributions (see [11]), also it has some advantages in terms of tractability, since it does not involve any special functions. The reliability function of the Kum-G distribution can be written in closed form, thus the Kum-G distribution can be used quite effectively even if the data are censored. Recently the composite between Kum and other distributions have been studied such as the Kum-Weibull by [12,13] and [14], Kum-log-logistic by [15], Kum generalized exponentiated Pareto by [16], also Kum Burr Type XII by [17], Kum modified Weibull by [18], exponentiated Kum-Dagum by [19] and Kum–Kum by [20].

In particular, if G has Burr III (c, k), with cdf given by

$$G(x;c,k) = (1+x^{-c})^{-k}, \qquad x > 0; \quad c,k > 0, \tag{4}$$

where c and k are shape parameters. Then the pdf corresponding to (4) is as follows

$$g(x;c,k) = ckx^{-(c+1)}(1+x^{-c})^{-(k+1)}, \qquad x>0; \quad c,k>0. \tag{5}$$

Substituting (4) in (2), the cdf of the Kum-BIII (a, b, c, k) distribution can be obtained as follows

$$F(x;a,b,c,k) = 1-(1-(1+x^{-c})^{-ak})^{b}, \qquad x>0; \quad a,b,c,k>0, \tag{6}$$

and the pdf corresponding to (6) is given by

$$f(x;\ a,b,\mathrm{c},\mathrm{k}) = abckx^{-(c+1)}(1+x^{-c})^{-(ak+1)}(1-(1+x^{-c})^{-ak})^{b-1}, \quad x>0; \ a,b,c,k>0. \tag{7}$$

The Kum-BIII distribution can be obtained as a special sub-model from the Beta-Burr III distribution which was introduced by [21] and exponentiated Kum-Dagum distribution which was introduced by [19].

The limiting distribution of the Kum-BIII (a,b,c,k) distribution, as the parameter c tends to infinity, is the cdf of Kum-I generalized logistic distribution among several others as special sub-models and the limiting distribution for the cdf of the Kum-BIII (a,b,c,k) distribution, given by (6), as the parameter k tends to infinity, is the cdf of the Kum-inverse Weibull distribution among several others as special sub-models.

The importance of the pdf in (7) is that it contains several well-known sub models distributions, such as the Kum-Burr XII, Kum-Burr II, Kum-Weibull, Kum-exponential, Kum-Rayleigh, Kum-Beta I, Kum-Beta II, Kum-Pareto II, Kum-I generalized logistic, Kum-extreme value, Kum-Gompertz and Kum-F distributions (see Table 1, in page 10). Clearly, the Burr III distribution is the basic exemplar for $a=b=1$. For $b=1$, it becomes the exponentiated Burr III distribution, which was introduced by [22].

1.1 A general expansion for the density function of the Kum-BIII distribution

The cdf given in (6) can be simplified using the binomial expansion theorem of the last bracket in the right hand side as shown below

$$F(x;a,b,c,k) = 1-ab\sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}\tau(x;\underline{\Theta}),$$

$$x>0;\ a,b,c,k>0, \tag{8}$$

where $b>0$ is non-integer, a is integer and $\tau(x;\underline{\Theta})$ is the Burr III cdf in (4) with parameters $\underline{\Theta}=(c,akj)$.

If $b>0$ is integer and a is integer, then the index j stops at b.

Similarly the pdf given in (7) can be written as follows:

$$f(x;a,b,c,k) = abck\ x^{-(c+1)}\sum_{j=0}^{\infty}w_j[1+x^{-c}]^{-s},$$

where $s=ak(j+1)+1$ and $w_j=(-1)^{j}\binom{b-1}{j}$, (9)

hence

$$f\left(x;\underline{\xi}\right) = ab\sum_{j=0}^{\infty}w_j\ g\left(x;\underline{\xi}\right), \qquad x>0; \quad a,b,c,k>0, \tag{10}$$

where

$b > 0$ is non-integer , a is integer and $g\left(x; \underline{\xi}\right)$ denotes the Burr III distribution with parameters $\underline{\xi} = (c, ak(j+1)+1)$.

If $b > 1$ is integer and *a* is integer then the index j stops at b – 1.

1.2 Reliability function

The reliability function (rf) for the Kum-BIII distribution is given below:

$$R1(x; a, b, c, k) = P(X \geq x) = (1 - (1 + x^{-c})^{-ak})^{b},$$

$$x > 0;\ a, b, c, k > 0, \tag{11}$$

where the limit of rf, as *k* or a tends to infinity, equals one.

Stress-strength reliability

The stress-strength model is a measure of the reliability of a component. Considering that *X* is a random strength of a component subjected to a random stress *Y*, a component fails if the applied stress is greater than the strength at any time and there is no failure when *Y* is less than *X*. Hence the reliability $R2$ is the probability that the unit is strong enough to overcome the stress and can be defined as follows

$$R2 = P(Y < X) = \int_x f_X(x)\, F_Y(x) dx \ ,$$

where $F_Y(x)$ is a cdf of Y at the point *x* and $f_X(x)$ is the pdf, *X* and *Y* are independent, also if X has the Kum-BIII ($a1$, *b1*, *c*, *k1)* and Y has the Kum-BIII ($a2$, *b2*, *c*, *k2*) as the parameters a, *b* and *k* change but *c* does not change.

Then

$$R2 = P(Y < X) = \int_0^\infty f(x;\ a1, b1, c, k1)\, F_y(x;\ a2, b2, c, k2) dx, \tag{12}$$

$$= \int_0^\infty a1b1ck1\, x^{-(c+1)}(1 + x^{-c})^{-(a1k1+1)}(1 - (1 + x^{-c})^{-a1k1})^{b1-1}$$

$$\times [1 - (1 - (1 + x^{-c})^{-a2k2})^{b2}]\, dx$$

$$= 1 - \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{a1b1k1(-1)^{l+m}\Gamma(b1-1)\Gamma(b2)}{m!l!(a1k1(m+1)+a2k2l)\, \Gamma(b1-m-1)\Gamma(b2-l)} \quad , \tag{13}$$

where $\Gamma(.)$ represents the gamma function.

The stress-strength modeling was studied by many authors [see [23] and [24]].

Hazard and reversed hazard functions

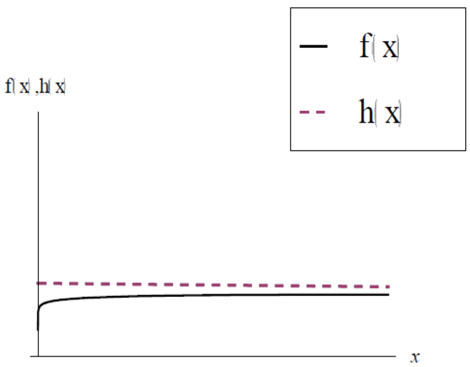
The hazard rate function (hrf) of the Kum-BIII distribution is given by

$$h(x; a, b, c, k) = \frac{f(x)}{R1(x)} = \frac{abckx^{-(c+1)}(1+x^{-c})^{-(ak+1)}}{1-(1+x^{-c})^{-ak}}, \ \ x > 0;\ a, b, c, k > 0, \tag{14}$$

and the reversed hazard (rh) rate function, which is also known by the dual of the hazard rate, extends the concept of the hazard rate to a reverse time direction and is given by

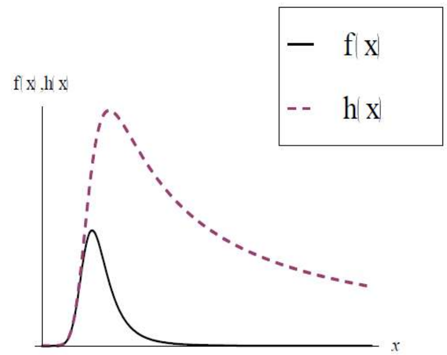
$$rh(x;a,b,c,k) = \frac{f(x)}{F(x)} = \frac{abckx^{-(c+1)}(1+x^{-c})^{-(ak+1)}\left(1-(1+x^{-c})^{-ak}\right)^{b-1}}{1-\left(1-(1+x^{-c})^{-ak}\right)^{b}}, \quad x>0; \quad a,b,c,k>0. \qquad (15)$$

1.3 Graphical description



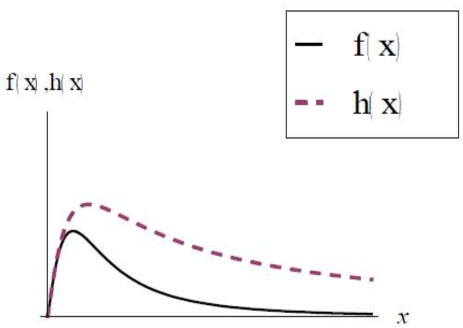
a =0.4, b=0.02, c=0.05, k=0.01

Figure 1



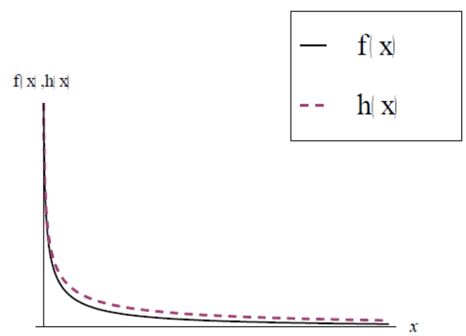
a =1.1, b=0.9, c=5, k=1.5

Figure 2



a =1.1, b=1.2, c=1.3, k=1.5

Figure 3



a =1.1, b=0.9, c=0.5, k=1.5

Figure 4

The plots of the probability density and hazard rate functions

From Figures 1-4, one can observe that the pdf is almost constant, approximately symmetric, decreasing and positive skewed respectively. The Kum-BIII distribution is a flexible model since the hrf represents major hazard shapes: constant, monotone decreasing and positive skewed respectively for different values of its parameters. In Figure 1, when the parameters *b*, *c* and *k* tend to zero, the hrf is approximately constant such as the hrf of the exponential distribution.

This paper is outlined as follows: In Section 2, some statistical properties are studied, such as: quantile function, skewness and kurtosis, order statistics and characteristic functions. In Section 3, some limiting distributions and relations between the Kum-BIII and other distributions are presented. Maximum likelihood estimation based on Type II censoring scheme is performed and the observed information matrix is determined in Section 4. In Section 5, Monte Carlo simulation is carried out to investigate the precision of the maximum likelihood estimates (MLEs) for different values of parameters and sample sizes. Finally some concluding remarks are given in Section 6.

2 Some Statistical Properties

In this section, the quantile function, the characteristic function, the r^{th} central and non-central moments, the mean, the variance, the skewness, the kurtosis and the i^{th} order statistic of the Kum-BIII (a, b, c, k) distribution are derived.

- The quantile function, can be obtained as follows:

$$x = F^{-1}(u) = \left[\left[1 - [1-u]^{1/b}\right]^{-1/ak} - 1\right]^{-1/c}, \quad 0 < u < 1. \tag{16}$$

One can easily generate x by taking u as a uniform random variable on $(0, 1)$.

Special quantiles can be obtained using (16). For example, if $u = 1/2$, the median of the Kum-BIII (a, b, c, k) is given by

median= $F^{-1}(1/2) = \left[\left[1 - [1-0.5]^{1/b}\right]^{-1/ak} - 1\right]^{-1/c}$.

- The characteristic function is

$$\varphi_x(t) = abk \sum_{j=0}^{\infty} w_j \sum_{d=0}^{\infty} \left(\frac{(\mathrm{i}t)^d}{d!}\right) \mathrm{B}\left(1 - \frac{d}{c}, s - 1 + \frac{d}{c}\right),$$
$$c > d, \; 1 + \frac{d}{c} < s. \tag{17}$$

where s and w_j are given by (9), B(., .) represents the beta function, i= $(-1)^{1/2}$ and $t \in$ R.

- The r^{th} non-central moments is as follows

$$\mu'_r = E(x^r) = abk \sum_{j=0}^{\infty} w_j \,\mathrm{B}\left(1 - \frac{r}{c}, s - 1 + \frac{r}{c}\right),$$
$$c > r, \; 1 + \frac{r}{c} < s. \tag{18}$$

The mean and the variance are given respectively by

$$\mu = abk \sum_{j=0}^{\infty} w_j \;\mathrm{B}\left(1 - \frac{1}{c}, s - 1 + \frac{1}{c}\right), \qquad c > 1, 1 + \frac{1}{c} < s,$$

and

$$V(x) = abk$$
$$\left[\sum_{j=0}^{\infty} w_j \,\mathrm{B}\left(1 - \frac{2}{c}, s - 1 + \frac{2}{c}\right) - abk\left(\sum_{j=0}^{\infty} w_j \,\mathrm{B}\left(1 - \frac{1}{c}, s - 1 + \frac{1}{c}\right)\right)^2\right],$$
$$c > 2, \; 1 + \frac{2}{c} < s,$$

where s and w_j are given by (9).

The central moments can be obtained using the relationship between the central moments and the non-central moments in (18) as follows:

$$\mu_r = \sum_{j=0}^{r} \binom{r}{j} (-1)^j \mu^j \mu'_{r-j} \ . \tag{19}$$

The standard moments can be obtained using (19), then

$$\alpha_r = \frac{\mu_r}{\sqrt{\mu_2{}^r}}. \tag{20}$$

When $r = 3$ in (20), then the skewness is given by

$$\alpha_3 = \frac{\left[\sum_{j=0}^{\infty} w_j B_3 - 3abk \sum_{j=0}^{\infty} w_j B_1 \sum_{j=0}^{\infty} w_j B_2 + 2(abk)^2 \left(\sum_{j=0}^{\infty} w_j B_1\right)^3\right]}{(abk)^{1/2}\left[\sum_{j=0}^{\infty} w_j B_2 - abk\left(\sum_{j=0}^{\infty} w_j B_1\right)^2\right]^{3/2}}, \tag{21}$$

where $B_l = \mathrm{B}\left(1 - \frac{l}{c}, s - 1 + \frac{l}{c}\right)$, $l = 1, 2, 3$ and $c > 3;\ 1 + \frac{l}{c} < s$.

When $r = 4$ in (20), then the kurtosis is

$$\alpha_4 = \frac{\left[\sum_{j=0}^{\infty} w_j B_4 - 4abk \sum_{j=0}^{\infty} w_j B_1 \sum_{j=0}^{\infty} w_j B_3 + 6\left(abk \sum_{j=0}^{\infty} w_j B_1\right)^2 \sum_{j=0}^{\infty} w_j B_2 - 3(abk)^3 \left(\sum_{j=0}^{\infty} w_j B_1\right)^4\right]}{abk\left[\sum_{j=0}^{\infty} w_j B_2 - abk\left(\sum_{j=0}^{\infty} w_j B_1\right)^2\right]^2}, \tag{22}$$

where $B_l = B\left(1 - \frac{l}{c}, s - 1 + \frac{l}{c}\right)$, $l = 1, 2, 3, 4$, $c > 4,\ 1 + \frac{l}{c} < s$,

s and w_j are given by (9).

Rényi entropy

An entropy of a random variable X with the pdf $f(.)$ is a measure of variation of the uncertainty and is denoted by $H_R(\rho)$. The Rényi entropy was introduced by [25]. It is defined by

$$H_R(\rho) = \frac{1}{1-\rho} \ln\left[\int_x (f(x))^\rho \, dx\right], \quad \rho > 0 \text{ and } \rho \neq 1. \tag{23}$$

Substituting (7) into (23), then the Rényi entropy of Kum-BIII (a, b, c, k) distribution is given by

$$H_R(\rho) = \frac{\rho}{1-\rho} (\ln a + \ln b + \ln c + \ln k)$$

$$+ \frac{1}{1-\rho} \ln \sum_{v=0}^{\infty} (-1)^v \binom{\rho(b-1)}{v} \left(-\frac{1}{c}\right) \mathrm{B}(v1, v2), \tag{24}$$

where $v1 = \rho\left(1 + \frac{1}{c}\right) - \frac{1}{c}$ and $v2 = ak(\rho + v) - \frac{1}{c}(\rho - 1)$.

The i^{th} order statistic

Let $X_1, X_2, \dots, X_n$ be independent identically distribution (iid) random variables from the Kum-BIII (a, b, c, k) distribution. Let $X_{(i)}$ denote the i^{th} order statistic. Then the pdf of $X_{(i)}$ can be written as a linear combination of Kum-BIII (a, b, c, k) density functions. It is well known that the pdf of i^{th} order statistic is given by

$$h_{i:n}\left(x_{(i)}\right) = D(i) f\left(x_{(i)}\right) F\left(x_{(i)}\right)^{i-1} \left[1 - F\left(x_{(i)}\right)\right]^{n-i}, \tag{25}$$

substituting the cdf and pdf given by (6) and (7) in (25), and using the binomial expansion theorem, then the pdf of $h_{i:n}(x_{(i)})$ can be rewritten as follows

$$h_{i:n}(x_{(i)}) = abkc\, D(i) \sum\nolimits_{(d_1,d_2,d_3)=0}^{*} x_{(i)}{}^{-\delta},$$

$$x > 0,\ \ a,b,c,k > 0,\ \ i = 1,2,\dots,n, \tag{26}$$

where

$$D(i) = i\binom{n}{i} = \frac{n!}{(i-1)!\,(n-i)!} = \frac{1}{B(i,n-i+1)},$$

$$\sum_{(d_1,d_2,d_3)=0}^{*} = \sum_{d_1=0}^{i-1}\sum_{d_2=0}^{\infty}\sum_{d_3=0}^{\infty}(-1)^{d_1+d_2+d_3}\binom{i-1}{d_1}\binom{z}{d_2}\binom{z_1}{d_3},$$

$\delta = c(d_3+1)+1, z = b(n-i+d_1+1)-1$ and $z_1 = ak(d_2+1)+1$.

Special cases:

I. The pdf of the first order statistic can be obtained; if $i = 1$ in (26), as follows

$$f_{1:n}(x_1) = \frac{n!}{(n-1)!}\, abkc \sum_{(g,g_1)=0}^{**} x_{(1)}{}^{-T},$$

$$x_{(1)} > 0;\ a,b,c,k > 0, \tag{27}$$

where

$$\sum\nolimits_{(g,g_1)=0}^{**} = \sum\nolimits_{g=0}^{\infty}\sum\nolimits_{g_1=0}^{\infty}(-1)^{g+g_1}\binom{z_2}{g}\binom{z_*}{g_1},$$

$T = c(g_1+1)+1,\ z_2 = bn-1$ and $z_* = ak(g+1)+1$.

II. The pdf of the largest order statistic can be obtained; if $i = n$ in (26), and is given by

$$f_{n:n}(x_n) = \frac{n!}{(n-1)!} abkc \sum_{(g_2,g_3,g_{3*})=0}^{***} x_{(n)}{}^{-T_1},$$

$$x_{(n)} > 0;\ \ a,b,c,k > 0, \tag{28}$$

where

$$\sum_{(g_2,g_3,g_{3*})=0}^{***} = \sum_{g_2=0}^{\infty}\sum_{g_3=0}^{\infty}\sum_{g_{3*}=0}^{\infty}(-1)^{g_2+g_3+g_{3*}}\binom{n-1}{g_2}\binom{z_3}{g_3}\binom{z_{3*}}{g_{3*}}$$

$T_1 = c(g_{3*}+1)+1,\ z_3 = b(1+g_2)-1$ and $z_{3*} = ak(g_3+1)+1$.

III. One can obtain the median observable in the odd case; if $i = \frac{n+1}{2}$ in (26), which is given by

$$f_{\frac{n+1}{2}:n}\left(x_{\left(\frac{n+1}{2}\right)}\right) = \frac{n!}{\left(\frac{n-1}{2}\right)!\left(\frac{n+1}{2}\right)!} abck \sum_{(g_4,g_5,g_{5*})=0}^{****} x_{\left(\frac{n+1}{2}\right)}^{-T_2} \quad ,$$

$$x_{\left(\frac{n+1}{2}\right)} > 0; \;\; a,b,c,k > 0, \qquad (29)$$

where

$$\sum_{(g_4,g_5,g_{5*})=0}^{****} = \sum_{g_4=0}^{\infty}\sum_{g_5=0}^{\infty}\sum_{g_{5*}=0}^{\infty} (-1)^{g_4+g_5+g_{5*}} \binom{z_4}{g_4}\binom{z_5}{g_5}\binom{z_{5*}}{g_{5*}},$$

$T_2 = c(g_{5*}+1)+1$, $z_5 = b\left(\frac{n+1}{2}\right) + g_4 - 1$ and $z_{5*} = ak(g_5+1)+1$.

Remark

All statistical properties of the Burr Type III (c,k) distribution, which was introduced by [5], can be derived from Kum-BIII (a,b,c,k) distribution, if $a = b = 1$.

3 Some Limiting and Transformed Distributions

Kum-BIII (a,b,c,k) distribution is related through variable transformations to a wide range of some other commonly distributions. In this section, some limiting and transformed distributions of the Kum-BIII (a,b,c,k) distribution are derived.

3.1 Some limiting distributions

The following limiting distributions provide relationships between Kum-BIII (a,b,c,k) and other well-known distributions.

I. If $X \sim$ Kum-BIII (a,b,c,k), then

$$\lim_{c\to\infty} P\left[X \le \exp\left(x\left(\frac{1+c}{c^2}\right)\right)\right] = 1 - [1 - [1+\exp(-x)]^{-ak}]^b,$$

$$-\infty < x < \infty; \;\; a,b,k > 0, \qquad (30)$$

which is the limit of the cdf of the Kum-BIII (a,b,c,k) distribution, given by (6), as the parameter c tends to infinity, and also is the cdf of a Kum-I generalized logistic distribution with parameters a, b and k. If k=1, the Kum-logistic distribution can be obtained. When $a = b = 1$, the Type I generalized logistic (k) distribution can be derived. For b = 1, it becomes the exponentiated Type I generalized logistic distribution. When a = b = k= *1*, the logistic distribution is given with mean = 0 and variance = $\pi^2/3$.

II. If $X \sim$ Kum-BIII (a,b,c,k), then

$$\lim_{k\to\infty} P\left[X \le \left(\frac{1}{k}\right)^{-1/c} x\right] = 1 - [1 - \exp(-ax^{-c})]^b \,,$$

$$x > 0, \;\; a,b,c > 0, \qquad (31)$$

which is the limit of the cdf of the Kum-BIII (a,b,c,k) distribution, given by (6), as the parameter k tends to infinity, which is the cdf of a Kum-inverse Weibull distribution with parameters a, b and c. If c =1, the

Kum-inverse gamma distribution can be obtained. When $a = b = 1$, the inverse Weibull distribution can be derived. For b = 1, the exponentiated inverse Weibull distribution can be proposed. When a = b = c= 1, inverse gamma distribution can be obtained.

3.2 Some transformed distributions

Table 1 presents the transformations of X which provide different relationships between Kum-BIII (a, b, c, k) and other well-known distributions such as Kum-Burr XII, Kum-Burr II, Kum-Weibull, Kum-exponential, Kum-Rayleigh, Kum-Beta I, Kum-Beta II, Kum-Pareto II, Kum-I generalized logistic, Kum-extreme value, Kum-Gompertz and Kum-F.

Table 1. Summary of transformations applied to the Kumaraswamy-Burr Type III and resulting distributions

Transformations	Distribution	pdf	Range
X^{-1}	Kum-Burr Type XII (a, b, c, k)	$abcky_1{}^{c-1}\,[1 + y_1{}^{c}]^{-(ak+1)}$ $\times [1 - [1 + y_1{}^{c}]^{-ak}]^{b-1}$	$y_1 > 0$
$c\ln\,(X)$	Kum-Burr Type II $(a, b,\ k)$	$abk\,\exp(-y_2)$ $\times [1 + \exp(-y_2)]^{-(ak+1)}$ $\times [1 - [1 + \exp(-y_2)]^{-ak}]^{b-1}$	$-\infty < y_2 < \infty$
$[\ln(1 + X^{-c})]^{1/c}$	Kum-Weibull (a, b, c, k)	$abcky_3{}^{c-1}$ $\times [\exp(-ky_3{}^{c})]^{a}$ $\times [1 - \exp(-ky_3{}^{c})^{a}]^{b-1}$	$y_3 > 0$
$\ln(1 + X^{-c})$	Kum-exponential(a, b, k)	$abk[\exp(-ky_4)]^{a}$ $\times\ [1 - [\exp(-ky_4)]^{a}]^{b-1}$	$y_4 > 0$
$[\ln(1 + X^{-c})]^{1/2}$	Kum- Rayleigh$(a, b, 2, k)$	$2abky_5[\exp(-ky_5{}^{2})]^{a}$ $\times [1 - \exp(-ky_5{}^{2})^{a}]^{b-1}$	$y_5 > 0$
$[1 + X^{-c}]^{-1}$	Kum-Beta Type I (ak, b)	$abky_6{}^{ak-1}[1 - y_6{}^{ak}]^{b-1}$	$0 < y_6 < 1$
X^{-c}	Kum-Beta Type II $(a, b, 1, k)$	$abk[1 + y_7]^{-(ak+1)}$ $\times\ [1 + (1 + y_7)^{-ak}]^{b-1}$	$y_7 > 0$
$1 + X^{-c}$	Kum-Pareto Type II $(a, b, 1, k)$	$abk(y_8)^{-(ak+1)}$ $\times [1 - (y_8)^{-ak}]^{b-1}$	$y_8 > 1$
$\ln(X^{-c})$	Kum-Type I generalized-logistic(a, b, k)	$abk\,\exp(-y_9)$ $\times\ [1 + \exp(-y_9)]^{-(ak+1)}$ $\times\ [1 - [1 + \exp(-y_9)]^{-ak}]^{b-1}$	$-\infty < y_9 < \infty$
$\ln[\ln(1 + X^{-c})^{k}]^{-1/k}$	Kum-extreme value (a, b, k)	abk $\times [\exp[-ky_{10} - a\,\exp(-ky_{10})]]$ $\times \left[1 - \left[\exp\left(\exp(-ky_{10})\right)\right]^{-a}\right]^{b-1}$	$-\infty < y_{10} < \infty$
$\ln[1 + \ln(1 + \mathrm{X}^{-c})]^{-1/u}$	Kum-Gompertz (a, b, u, z)	$abz\left[\exp\left[uy_{11} + \left[\exp\left(-\frac{z}{u}(uy_{11} - 1)\right)\right]^{a}\right]\right]$ $\times \left[1 - \left[\exp\left(-\frac{z}{u}[\exp(uy_{11} - 1)]\right)\right]^{a}\right]^{b-1}$	$y_{11} > 0$
kX^{-c}	Kum-F $(a, b, 2, 2k)$	$ab\left[1 + \frac{y_{12}}{k}\right]^{-(ak+1)}$ $\times \left[1 - \left(1 + \frac{y_{12}}{k}\right)^{-ak}\right]^{b-1}$	$y_{12} > 0$

4 Maximum Likelihood Estimation

Suppose that $X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(r)}$ is a censored sample of size r obtained from a life-test on n items (Type II censored sample) whose lifetimes have a Kum-BIII (a, b, c, k) distribution. The likelihood function in this case is given by

$$L(\underline{\theta}|\underline{x}) = \frac{n!}{(n-r)!}\{\prod_{i=1}^{r} f(x_{(i)}; \underline{\theta})\}[R(x_{(r)}; \underline{\theta})]^{n-r}, \tag{32}$$

where $\underline{\theta} = (a, b, c, k)'$, $f(x_{(i)}; \underline{\theta})$ and $R(x_{(r)}; \underline{\theta})$ are given by (7) and (11), respectively.

The natural logarithm of $L(\underline{\theta}|\underline{x})$is given by

$$\ell \equiv \ln L(\underline{\theta}; \underline{x}) \propto \sum_{i=1}^{r} \ln f(x_{(i)}; \underline{\theta}) + (n-r) \ln R(x_{(r)}; \underline{\theta}), \tag{33}$$

and substituting (7) and (11) in (33) yields

$$\ell \propto r ln(a) + r\ln(b) + r\ln(c) + r\ln(k) - c\sum_{i=1}^{r}\ln(x_{(i)})$$
$$-(ak+1)\sum_{i=1}^{r}\ln(z_i) + (b-1)\sum_{i=1}^{r}\ln(1 - z_i^{-ak}) + b(n-r)\ln(1 - z_\circ^{-ak}), \tag{34}$$

where .

$$\begin{cases} z_i = (1 + x_{(i)}^{-c}), \\ z_\circ = (1 + x_{(r)}^{-c}). \end{cases} \tag{35}$$

The maximum likelihood estimators (MLEs) of $\underline{\theta}$ can be derived by differentiating ℓ in (34) with respect to a, b, c and k and then setting to zero as given bellow,

$$\frac{\partial \ell}{\partial a} = \frac{r}{a} - k\sum_{i=1}^{r}\ln(z_i) + k(b-1)\sum_{i=1}^{r}\frac{(z_i^{-ak})(\ln(z_i))}{(1 - z_i^{-ak})}$$
$$+kb(n-r)\left[\frac{(z_\circ^{-ak})(\ln(z_\circ))}{(1 - z_\circ^{-ak})}\right], \tag{36}$$

$$\frac{\partial \ell}{\partial b} = \frac{r}{b} + \sum_{i=1}^{r}\ln(1 - z_i^{-ak}) + (n-r)\ln(1 - z_\circ^{-ak}), \tag{37}$$

$$\frac{\partial \ell}{\partial c} = \frac{r}{c} - \sum_{i=1}^{r}\ln(x_{(i)}) + \left[(ak+1)\sum_{i=1}^{r}\frac{(x_i^{-c})(\ln(x_{(i)}))}{z_i}\right]$$
$$-\left[ak(b-1)\sum_{i=1}^{r}\frac{(z_i^{-(ak+1)})(x_i^{-c}\ln(x_{(i)}))}{(1 - z_i^{-ak})}\right]$$
$$-ak\, b(n-r)\left[\frac{(z_\circ^{-(ak+1)})(x_r^{-c}\ln(x_{(r)}))}{(1 - z_\circ^{-ak})}\right], \tag{38}$$

and

$$\frac{\partial \ell}{\partial k} = \frac{r}{k} - a\sum_{i=1}^{r}\ln(z_i) + \left[a(b-1)\sum_{i=1}^{r}\frac{(z_i{}^{-ak})(\ln(z_i))}{(1-z_i{}^{-ak})}\right]$$

$$+ ab(n-r)\left[\frac{(z_\circ{}^{-ak})(\ln(z_\circ))}{(1-z_\circ{}^{-ak})}\right]. \quad (39)$$

The solution of the system of nonlinear Equations (36), (38) and (39) can be solved numerically to obtain the MLEs of the parameters a, c, k and the MLE of b can be obtained by equating (37) to zero, hence

$$\hat{b} = \frac{-r}{s_1}, \quad (40)$$

where

$$s_1 = \sum_{i=1}^{r}\ln(1-z_i{}^{-ak}) + (n-r)\ln(1-z_\circ{}^{-ak}).$$

Remarks:

I. If $b^{-1} = \vartheta$, then from (37) the uniformly minimum variance unbiased estimator (UMVUE) of ϑ is,

$$\hat{\vartheta} = \frac{-1}{r}\left[\sum_{i=1}^{r}\ln(1-z_i{}^{-ak}) + (n-r)\ln(1-z_\circ{}^{-ak})\right], \quad (41)$$

where z_i and $z_\circ$ are defined by (35).

II. When $r = n$, all the results obtained for Type II censored sample reduce to those of the complete sample.

III. Considering that $X_1, X_2, \ldots, X_n$ is a random sample of size n drawn from a Kum-BIII (a, b, c, k) distribution with pdf given by (7). One can obtain a sufficient and complete statistic for the parameter b using the exponential family which is s_2,

$$s_2 = \sum_{i=1}^{n}\ln[1-(1+x^{-c})^{-ak}]. \quad (42)$$

Maximum likelihood estimators for the reliability and hazard rate functions

Applying the invariance property, the MLEs of the rf and hrf are obtained by replacing the parameters a, b, c and k in (11) and (14) by their MLEs.

Hence, for a given value of x, the MLEs of $R(x)$ and $h(x)$ are given, respectively by

$$\hat{R}(x) = \left(1-\left(1+x^{-\hat{c}}\right)^{-\hat{a}\hat{k}}\right)^{\hat{b}-1}, \qquad x > 0, \quad (43)$$

and

$$\hat{h}(x) = \frac{\hat{a}\hat{b}\hat{c}\hat{k}x^{-(\hat{c}+1)}\left(1+x^{-\hat{c}}\right)^{-(\hat{a}\hat{k}+1)}}{1-\left(1+x^{-\hat{c}}\right)^{-\hat{a}\hat{k}}}, \qquad x > 0, \quad (44)$$

where $\hat{a}, \hat{b}, \hat{c}$ *and* $\hat{k}$ are the MLEs of a, b, c and k.

The asymptotic Fisher information matrix is given by

$$\tilde{I} = -\left[\frac{\partial^2 \ell}{\partial\theta_i \partial\theta_j}\right], \qquad i,j = 1,2,3,4, \tag{45}$$

where $\theta_1 = a$, $\theta_2 = b$, $\theta_3 = c$ and $\theta_4 = k$ and the elements of the information matrix are derived.

The asymptotic variance-covariance matrix of the MLEs $\hat{a}, \hat{b}, \hat{c}$ and $\hat{k}$ is the inverse of the asymptotic Fisher information matrix.

For large sample size, the MLEs under regularity conditions are consistent and asymptotically unbiased as well as asymptotically normally distributed, hence the asymptotic confidence intervals (ACI) for the MLE is obtained by $P\left(-Z < \frac{\hat{\theta}_{iML} - \theta_i}{\sigma_{\hat{\theta}_{iML}}} < Z\right) = 1 - \alpha$ where Z is the $100\left(1 - \frac{\alpha}{2}\right)$th standard normal percentile. The two sided approximate $100(1 - \alpha)\%$ the confidence intervals are as shown below

$$LL_\theta = \hat{\theta}_{iML} - Z_{\frac{\alpha}{2}} \hat{\sigma}_{\hat{\theta}_{iML}} \quad \text{and} \quad UL_\theta = \hat{\theta}_{iML} + Z_{\frac{\alpha}{2}} \hat{\sigma}_{\hat{\theta}_{iML}} \ , \tag{46}$$

where $\hat{\sigma}_{\hat{\theta}_{iML}}$ is the standard deviation and $\hat{\theta}_{iML}$ is $\hat{a}$, $\hat{b}$, $\hat{c}$ or $\hat{k}$ respectively.

5 Simulation Study

- In this section, a numerical study is given to illustrate the results obtained on the basis of generated data from Kum-BIII (a, b, c, k) distribution.
- The computations are performed using Mathematica 9, where N = 1000 is the number of repetitions, for different sample sizes (n=20, 50, 100) and the number of survival units are (r= 0.90 n and 0.80n).
- Tables 2 and 3, in Appendix 1, show the MLEs of the parameters, rf and hrf where the initial parameter values are a =1.1, b=1.2, c=1.3 and k=1.5 based on two levels of Type II censoring. Similarly Tables 5 and 6 display the MLEs of the parameters, rf and hrf with the initial parameter values a =0.7, b=0.9, c=1.2 and k=1.4.
- Some measurements of accuracy are used to evaluate the performance of the estimators a, b, c and k. Tables 2 and 5 show the variances, biases^2 and the estimated risks (ER) of the estimates to study the precision and the variation of MLEs. Also Tables 4 and 6 present the estimated risks of the reliability and the hazard rate functions.
- Tables 4 and 7 present the two-sided 95% ACI for the parameters, rf and hrf of Kum-BIII (a, b, c, k) distribution. These tables contain the estimates, lower limit (LL), upper limit (UL) and the length of the intervals.

6 Concluding Remarks

- Tables 2 and 5 indicate that the variances, biases^2 and ER decrease when the sample size n increases. For all sample sizes Table 2 shows that, the ER ($\hat{a}$) performs better than other estimates. It is observed that as the level of censoring decreases the variances, bias^2 and ER decrease.
- Tables 3 and 6 show that the ER of rf and hrf decrease when the different values of time t_0 and the sample size increase, while the hrf increases when the different values of time t_0 and the sample size increase.
- From Tables 4 and 7, for all different sample sizes, one can observe that the lengths of the ACI of the four model parameters, rf and hrf become narrower when the sample size n increases and the level of censoring decreases. Also the lengths of $\hat{a}$ performs better than other estimates.

- The first and the last remark are expected since decreasing the level of censoring means that more information is provided by the sample and hence increases the accuracy of the estimates.

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Competing Interests

Authors have declared that no competing interests exist.

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Appendix 1

Table 2. ML estimates of the parameters, variances, biases^2 and their estimated risks based on Type II censoring, *N*=1000, *r=0.80 n* and *0.90 n*, a =1.1, *b*=1.2, *c*=1.3 and *k*=1.5

n	*r*	Estimates	Variance	$bias^2$	ER
20	16	$\hat{a}$	0.0409	0.0045	0.0455
		$\hat{b}$	0.2087	0.0136	0.2223
		$\hat{c}$	0.1309	0.0274	0.1583
		$\hat{k}$	0.0729	0.0031	0.0760
	18	$\hat{a}$	0.0313	0.0023	0.0336
		$\hat{b}$	0.1554	0.0043	0.1598
		$\hat{c}$	0.0785	0.0212	0.0997
		$\hat{k}$	0.0556	0.0009	0.0556
50	40	$\hat{a}$	0.0288	0.0011	0.0299
		$\hat{b}$	0.1589	0.0003	0.1593
		$\hat{c}$	0.0593	0.0313	0.0906
		$\hat{k}$	0.0512	0.0001	0.0513
	45	$\hat{a}$	0.0193	0.0001	0.0194
		$\hat{b}$	0.0760	0.0038	0.0799
		$\hat{c}$	0.0398	0.0259	0.0658
		$\hat{k}$	0.0343	0.0004	0.0347
100	80	$\hat{a}$	0.0195	0.0000	0.0195
		$\hat{b}$	00836	0.0198	0.1034
		$\hat{c}$	0.0428	0.0432	0.0860
		$\hat{k}$	0.0347	0.0018	0.0364
	90	$\hat{a}$	0.0095	0.0001	0.0096
		$\hat{b}$	0.0313	0.0139	0.0453
		$\hat{c}$	0.0255	0.0269	0.0525
		$\hat{k}$	0.0168	0.0025	0.0193

Table 3. The estimated reliability and hazard rate functions at different time t_0 and different sample sizes

n	*r*	t_0	$\hat{R}(t_0)$	ER	$\hat{h}(t_0)$	ER
	16	0.4	0.9048	0.0059	0.4017	0.6572
		1	0.6486	0.0166	0.6803	0.1015
20	18	0.4	0.9042	0.0040	0.4487	0.0592
		1	0.6445	0.0138	0.6649	0.0717
	40	0.4	0.9072	0.0028	0.4299	0.0998
		1	0.6592	0.0085	0.6224	0.0266
50	45	0.4	0.9049	0.0019	0.4566	0.0269
		1	0.6471	0.0064	0.6283	0.0196
	80	0.4	0.9131	0.0015	0.4215	0.0204
		1	0.6592	0.0047	0.6059	0.0101
100	90	0.4	0.9063	0.0011	0.4525	0.0137
		1	0.6446	0.0031	0.6218	0.0094

Table 4. Confidence intervals for the parameters a, b, c, k, rf and hrf based on Type II censoring at confidence level 95% at different sample sizes

n	*r*	Parameters	Estimates	*UL*	*LL*	Length
20	16	a	1.1671	1.5639	0.7703	0.7936
		b	1.3167	2.2121	0.4213	1.7908
		c	1.4656	2.1747	0.7564	1.4183
		k	1.5561	2.0852	1.0270	1.0582
		$R(t)$	0.6486	0.8989	0.3982	0.5008
		$h(t)$	0.6803	1.2855	0.0751	1.2104
	18	a	1.1477	1.4944	0.8009	0.6934
		b	1.2659	2.0386	0.4932	1.5454
		c	1.4456	1.9947	0.8965	1.0982
		k	1.5302	1.9925	1.0679	0.9246
		$R(t)$	0.6445	0.8733	0.4157	0.4576
		$h(t)$	0.6649	1.1749	0.1548	1.0201
50	40	a	1.1329	1.4655	0.8004	0.6649
		b	1.1818	1.9633	0.4003	1.5629
		c	1.4769	1.9544	0.9994	0.9549
		k	1.5106	1.9539	1.0673	0.8866
		$R(t)$	0.6592	0.8309	0.4875	0.3434
		$h(t)$	0.6224	0.9394	0.3054	0.6341
	45	a	1.1097	1.3818	0.8377	0.5442
		b	1.1380	1.6785	0.5975	1.0810
		c	1.4612	1.8522	1.0701	0.7820
		k	1.4797	1.8424	1.1169	0.7256
		$R(t)$	0.6471	0.8013	0.4930	0.3082
		$h(t)$	0.6283	0.8979	0.3588	0.5391
100	80	a	1.0936	1.3673	0.8199	0.5474
		b	1.0594	1.6262	0.4925	1.1338
		c	1.5078	1.9135	1.1022	0.8113
		k	1.4581	1.8230	1.0932	0.7298
		$R(t)$	0.6592	0.7824	0.5360	0.2463
		$h(t)$	0.6059	0.8023	0.4096	0.3928
	90	a	1.0878	1.2785	0.8972	0.3813
		b	1.0817	1.4283	0.7351	0.6932
		c	1.4641	1.7774	1.1509	0.6265
		k	1.4504	1.7046	1.1962	0.5084
		$R(t)$	0.6446	0.7506	0.5385	0.2121
		$h(t)$	0.6218	0.8081	0.4355	0.3726

Table 5. ML estimates of the parameters, variances, biases^2 and their estimated risks based on Type II censoring, *N*=1000, *r=0.80 n* and *0.90 n*, a =0.7, *b*=0.9, *c*=1.2 and *k*=1.4

n	*r*	estimates	variance	$bias^2$	ER
20	16	$\hat{a}$	0.0185	0.0318	0.0503
		$\hat{b}$	0.1153	0.0178	0.1331
		$\hat{c}$	0.0988	0.0030	0.1018
		$\hat{k}$	0.0391	0.0149	0.0540
	18	$\hat{a}$	0.0143	0.0278	0.0421
		$\hat{b}$	0.0858	0.0078	0.0934
		$\hat{c}$	0.0651	0.0033	0.0680
		$\hat{k}$	0.0303	0.0194	0.0497
50	40	$\hat{a}$	0.0057	0.0184	0.0241
		$\hat{b}$	0.0454	0.0029	0.0483
		$\hat{c}$	0.0217	0.0072	0.0289
		$\hat{k}$	0.0121	0.0341	0.0461
	45	$\hat{a}$	0.0044	0.0184	0.0228
		$\hat{b}$	0.0315	0.0012	0.0328
		$\hat{c}$	0.0122	0.0043	0.0165
		$\hat{k}$	0.0093	0.0340	0.0433
100	80	$\hat{a}$	0.0018	0.0144	0.0162
		$\hat{b}$	0.0191	0.0001	0.0192
		$\hat{c}$	0.0062	0.0087	0.0149
		$\hat{k}$	0.0038	0.0429	0.0467
	90	$\hat{a}$	0.0002	0.0140	0.0142
		$\hat{b}$	0.0112	6.0553×10^{-8}	0.0112
		$\hat{c}$	0.0050	0.0057	0.0107
		$\hat{k}$	0.0040	0.0403	0.0443

Table 6. The estimated reliability and hazard rate functions at different time t_0 and different sample sizes

n	*r*	t_0	$\hat{R}(t_0)$	ER	$\hat{h}(t_0)$	ER
	16	0.2	0.8884	0.0081	0.5421	2.1067
		0.4	0.7816	0.0095	0.6634	0.1995
20	18	0.2	0.8936	0.0051	0.5962	1.3250
		0.4	0.7843	0.0078	0.6710	0.0662
	40	0.2	0.8892	0.0023	0.6680	0.3002
		0.4	0.7766	0.0037	0.7069	0.0286
50	45	0.2	0.8927	0.0415	0.6648	0.0361
		0.4	0.7769	0.0036	0.6975	0.0267
	80	0.2	0.8926	0.0009	0.6736	0.0196
		0.4	0.7764	0.0019	0.7036	0.0172
100	90	0.2	0.8917	0.0008	0.6743	0.0169
		0.4	0.7788	0.0013	0.6891	0.0135

Table 7. Confidence intervals for the parameters a, b, c, k, rf and hrf based on Type II censoring at confidence level 95% at different sample sizes

n	*r*	parameters	estimates	*UL*	*LL*	Length
20	16	a	0.8784	1.1447	0.6120	0.5327
		b	1.0335	1.6991	0.3679	1.3311
		c	1.2549	1.8709	0.6389	1.2319
		k	1.2776	1.6651	0.8902	0.7749
		$R(t)$	0.7816	0.9697	0.5936	0.3761
		$h(t)$	0.6634	1.5377	0	1.5377
	18	a	0.8668	1.1013	0.6323	0.4689
		b	0.9875	1.5616	0.4134	1.1481
		c	1.2578	1.7577	0.7579	0.9999
		k	1.2608	1.6018	0.9198	0.6821
		$R(t)$	0.7843	0.9531	0.6155	0.3376
		$h(t)$	0.6710	1.1745	0.1676	1.0069
50	40	a	0.8356	0.9837	0.6875	0.2961
		b	0.9536	1.3713	0.5358	0.8354
		c	1.2849	1.5736	0.9963	0.5773
		k	1.2154	1.4308	1.0001	0.4307
		$R(t)$	0.7766	0.8944	0.6587	0.2356
		$h(t)$	0.7069	1.0357	0.3783	0.6574
	45	a	0.8357	0.9656	0.7057	0.2599
		b	0.9354	1.2835	0.5872	0.6962
		c	1.2653	1.4818	1.0487	0.4331
		k	1.2155	1.4046	1.0265	0.3781
		$R(t)$	0.7769	0.8926	0.6612	0.2313
		$h(t)$	0.6975	1.0173	0.3777	0.6396
100	80	a	0.8202	0.9032	0.7371	0.1661
		b	0.9101	1.1813	0.6389	0.5423
		c	1.2934	1.4472	1.1395	0.3078
		k	1.1929	1.313	1.0721	0.2416
		$R(t)$	0.7764	0.8607	0.6920	0.1687
		$h(t)$	0.7036	0.9583	0.4488	0.5095
	90	a	0.8245	0.9001	0.7370	0.1631
		b	0.8998	1.1069	0.6926	0.4144
		c	1.2756	1.4144	1.1367	0.2777
		k	1.1993	1.3237	1.0749	0.2488
		$R(t)$	0.7789	0.8587	0.6988	0.1599
		$h(t)$	0.6891	0.9167	0.4615	0.4553

Appendix 2

The elements of the information matrix; the second partial derivatives of the log-likelihood function, are given below

$$I_{aa} = \frac{\partial^2 \ell}{\partial a^2} = \frac{-r}{a^2} + (b-1)\sum_{i=1}^{r} \frac{(-)(1-z_i{}^{-ak})z_i{}^{-ak}(k\ln(z_i))^2 - [z_i{}^{-ak}(k\ln(z_i))]^2}{[1-z_i{}^{-ak}]^2}$$

$$+b(n-r)\left(\frac{(-)(z_\circ{}^{-ak})(k\ln(z_\circ))^2\cdot(1-z_\circ{}^{-ak}) - \left[(z_\circ{}^{-ak})(k\ln(z_\circ))\right]^2}{\left(1-z_\circ{}^{-ak}\right)^2}\right),$$

$$I_{bb} = \frac{\partial^2 \ell}{\partial b^2} = \frac{-r}{b^2} \ ,$$

$$I_{cc} = \frac{\partial^2 \ell}{\partial c^2} = \frac{-r}{c^2} - \left[(ak+1)\sum_{i=1}^{r}\frac{(-)z_i x_i{}^{-c}\ln(x_{(i)})^2 + \left(x_i{}^{-c}\ln(x_{(i)})\right)^2}{[z_i]^2}\right] - (b-1)$$

$$\times \{\sum_{i=1}^{r} \frac{\left(1-z_i{}^{-ak}\right)\left[\left(ak(ak+1)z_i{}^{-(ak+2)}\right)\left(x_i{}^{-c}\ln(x_{(i)})\right)^2 - \left(akx_i{}^{-c}z_i{}^{-(ak+1)}\left(\ln(x_{(i)})\right)^2\right)\right]}{\left(1-z_i{}^{-ak}\right)^2}$$

$$+\frac{\left(akx_i{}^{-c}\ln\left(x_{(i)}\right)z_i{}^{-(ak+1)}\right)^2}{\left(1-z_i{}^{-ak}\right)^2}\} - b(n-r)[\frac{\left(1-z_\circ{}^{-ak}\right)\{-(ak)(z_\circ{}^{-(ak+1)})\left(x_r{}^{-c}\langle\ln\left(x_{(r)}\right)\rangle^2\right)}{\left(1-z_\circ{}^{-ak}\right)^2}$$

$$+\frac{(ak)(ak+1)z_\circ{}^{-(ak+2)}\left(x_r{}^{-c}\ln\left(x_{(r)}\right)\right)^2\} + \left[(ak)(z_\circ{}^{-(ak+1)})\left(x_r{}^{-c}\ln\left(x_{(r)}\right)\right)\right]^2}{\left(1-z_\circ{}^{-ak}\right)^2}],$$

$$I_{kk} = \frac{\partial^2 \ell}{\partial k^2} = \frac{-r}{k^2}$$

$$+(b-1)\sum_{i=1}^{r}\frac{-[(1-z_i{}^{-ak})z_i{}^{-ak}(a\ln(z_i))^2] - [z_i{}^{-ak}(a\ln(z_i))]^2}{[1-z_i{}^{-ak}]^2}$$

$$+b(n-r)[\frac{-\left(z_\circ{}^{-ak}a\ln(z_\circ)\right)^2 - (1-z_\circ{}^{-ak})\left(z_\circ{}^{-ak}(a\ln(z_\circ))^2\right)}{\left(1-z_\circ{}^{-ak}\right)^2}],$$

$$I_{ab} = \frac{\partial^2 \ell}{\partial a \partial b} = \sum_{i=1}^{r}\frac{z_i{}^{-ak}(k\ln(z_i))}{(1-z_i{}^{-ak})} + (n-r)\left[\frac{(z_\circ{}^{-ak})(k\ln(z_\circ))}{(1-z_\circ{}^{-ak})}\right],$$

$$I_{ac} = \frac{\partial^2 \ell}{\partial a \partial c} = -k\sum_{i=1}^{r}\frac{-x_i{}^{-c}\ln(x_{(i)})}{z_i} + (b-1)$$

$$\times [\sum_{i=1}^{r} \frac{\left(1-z_i{}^{-ak}\right)\left[(ak\, z_i{}^{-(ak+1)})x_i{}^{-c}\ln(x_{(i)})(k\ln(z_i)) - kz_i{}^{-ak}\left(\frac{x_i{}^{-c}\ln(x_{(i)})}{z_i}\right)\right]}{\left[1-z_i{}^{-ak}\right]^2}$$

$$+\frac{(ak\,z_i^{-(ak+1)})x_i^{-c}\ln(x_{(i)})(z_i^{-ak})(k\ln(z_i))}{\left[1-z_i^{-ak}\right]^2}] + b(n-r)\{\frac{(1-z_\circ^{-ak})\{-z_\circ^{-ak}\left(\frac{kx_r^{-c}\ln(x_{(r)})}{z_\circ}\right)}{(1-z_\circ^{-ak})^2}$$

$$+\frac{ak^2(z_\circ^{-(ak+1)})(x_r^{-c}\ln(x_{(r)}))\ln(z_\circ)\}+ak^2z_\circ^{-(2ak+1)}(x_r^{-c}\ln(x_{(r)}))\ln(z_\circ)}{(1-z_\circ^{-ak})^2}\},$$

$$I_{ak}=\frac{\partial^2\ell}{\partial a\partial k}=-\sum_{i=1}^{r}\ln(z_i)+(b-1)$$

$$\times\left[\sum_{i=1}^{r}\frac{(1-z_i^{-ak})\left[z_i^{-ak}\ln(z_i)-\{z_i^{-ak}(a\ln(z_i))(k\ln(z_i))\}\right]-\left[(z_i^{-ak})^2(a\ln(z_i))(k\ln(z_i))\right]}{\left[1-z_i^{-ak}\right]^2}\right]$$

$$+b(n-r)[\frac{(1-z_\circ^{-ak})\{z_\circ^{-ak}\ln(z_\circ)-akz_\circ^{-ak}(\ln(z_\circ))^2\}-ak(z_\circ^{-ak}\ln(z_\circ))^2}{(1-z_\circ^{-ak})^2}],$$

$$I_{bc}=\frac{\partial^2\ell}{\partial b\partial c}=\sum_{i=1}^{r}\frac{-(ak\,z_i^{-(ak+1)})x_i^{-c}\ln(x_{(i)})}{(1-z_i^{-ak})}-(n-r)\frac{akz_\circ^{-(ak+1)}(x_r^{-c}\ln(x_{(r)}))}{1-z_\circ^{-ak}},$$

$$I_{bk}=\frac{\partial^2\ell}{\partial b\partial k}=\sum_{i=1}^{r}\frac{z_i^{-ak}(a\ln(z_i))}{(1-z_i^{-ak})}+(n-r)\frac{az_\circ^{-ak}\ln(z_\circ)}{1-z_\circ^{-ak}}$$

and

$$I_{kc}=\frac{\partial^2\ell}{\partial k\partial c}=a\sum_{i=1}^{r}\frac{x_i^{-c}\ln(x_{(i)})}{z_i}+(b-1)$$

$$\times\{\sum_{i=1}^{r}\frac{(1-z_i^{-ak})\left[(ak\,z_i^{-(ak+1)})x_i^{-c}\ln(x_{(i)})(a\ln(z_i))-az_i^{-ak}\left(\frac{x_i^{-c}\ln(x_{(i)})}{z_i}\right)\right]}{\left[1-z_i^{-ak}\right]^2}$$

$$+\frac{\{(ak\,z_i^{-(ak+1)})x_i^{-c}\ln(x_{(i)})(z_i^{-ak})(a\ln(z_i))\}}{\left[1-z_i^{-ak}\right]^2}.$$

$$+b(n-r)[\frac{(1-z_\circ^{-ak})\{a^2k\,z_\circ^{-(ak+1)}\ln(z_\circ)(x_r^{-c}\ln(x_{(r)}))-az_\circ^{-ak}\left(\frac{x_r^{-c}\ln(x_{(r)})}{z_\circ}\right)\}}{(1-z_\circ^{-ak})^2}$$

$$+\frac{a^2k\,z_\circ^{-(2ak+1)}\ln(z_\circ)(x_r^{-c}\ln(x_{(r)}))}{(1-z_\circ^{-ak})^2}].$$

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