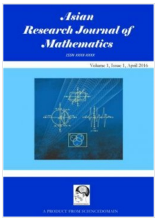


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# Dynamic Response of Non-prismatic Bernoulli Euler Beam with Exponentially Varying Thickness Resting on Variable Elastic Foundation

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***Author's contribution***

*This work was carried out in collaboration between both authors. Author JSA designed the study, performed the statistical analysis, wrote the protocol and wrote the first draft of the manuscript. Author JSA and AA managed the analyses of the study. Both authors read and approved the final manuscript.*

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## Abstract

In this paper, the motion of a non-prismatic Bernoulli-Euler beam with exponentially varying thickness resting on variable elastic foundation and under the loads moving with constant velocity is analyzed. The governing equation is a fourth order partial differential equation. The solution technique is based on the method of Galerkin with series representation of Heaviside function and Struble's asymptotic method. The results shows that, for the same natural frequency, the critical speed for the system traversed by moving force is greater than that under the influence of a moving mass. Also, increase in the values of the structural parameters such as foundation stiffness, foundation modulus, length of the beam and exponential factor reduces the response

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amplitude of the beam for both moving force and moving mass problems. Furthermore, it is found that the moving force solution is not always an upper bound for the accurate solution for the non-prismatic Bernoulli-Euler beam.

*Keywords: Non-prismatic beam, variable elastic foundation, exponentially varying thickness, Struble's asymptotic method.*

# 1 Introduction

The study of non-prismatic beam under moving loads forms a very important structural element in engineering design and construction. It has also become the objective of various researchers in the field of Applied Mathematics. In general, problems of this type are mathematically complex if analytical approach is used. Thus, most of the research works available in the Literature are those in which Numerical technique is used. This is due to great amount of computational labor which is required both to set up and solve the necessary equations. A major break-through in this field of research is the work of Timoshenko [1] who gave impetus to research work in this area of study. The analysis of the dynamic response of a simple beam continuously supported by a viscoelastic foundation to a moving load, moving at variable speed was considered. The analysis reveals several resonance conditions depending on the viscoelasticity of the foundation. Also a theory for the response to an arbitrary number of concentrated moving masses of a rectangular plate continuously supported by an elastic Pasternak type foundation was developed [2]. It was found that the critical speeds of the system increased with increase in the values of the foundation modulli whether the inertia of the moving load is considered or not. The displacement response of a simply supported non-uniform beam resting on an elastic foundation to several moving load was later taken up and concluded that the maximum transverse deflection of the beam is always greater than the displacement of the moving mass [3]. A modification of the asymptotic method was used to simplify the resulting sequence of differential equation [4]. Chen employed the differential quadrature element method (DQEM), to investigate the vibration of beams on Winkler [5] and Pasternak [6] foundations. Hosing et al. [7] equally worked on the solution to natural flexural vibrations of a continuous beam on discrete elastic support . DTM, first proposed by Zhou [8], was employed to find free vibration of a constant thickness beam on elastic soil by Catal [9] and Balkaya et al. [10] . Other than the above, in recent years, a few researches have been conducted concentrating on exponential characteristic of beams. Awodola [11] solve the problem of vibration of a beam under exponentially varying magnitude moving load. Mao and Pietrzko [12] used the Adomian decomposition method (ADM) to examine the free vibration of a beam with a continuously exponential variation of width and a constant thickness along the length. Sayad et al. Study vibration analyses of a tapered beam with exponentially varying thickness resting on Winkler foundation using the differential transform method [13]. It is well known that in the dynamical system like this, analytical are desirable as a method of solution, as there often shed light on vital information about the vibrating system.

Thus, this paper studied the flexural motions under moving loads of non-prismatic Bernoulli-Euler beam with exponentially varying thickness resting on variable elastic foundation using analytical approach. Several numerical examples will also be presented. It is assumed that the speed at which the load traverses the structural element is constant.

# 2 Methodology

The motion of a non-prismatic Bernoulli-Euler beam with exponentially varying thickness resting on variable elastic foundation and under the of loads moving with constant velocity is governed by the following fourth order partial differential equation

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2}{\partial x^2}V(x,t)\right]+\rho A(x)\frac{\partial^2 V}{\partial t^2}+\ K(x)\,V(x,t)-\ G(x)\frac{\partial^2}{\partial x^2}\ V(x,t)=f(x,t) \qquad (2.1)$$

Where *x (0 ≤ x ≤ L)* is the distance along the beam; *t* is the time; *I(x)* is the variable moment of inertia of the beam cross section at a distance *x;* $\boldsymbol{\rho}$ is the beam mass per unit volume; *E* is the beam elastic modulus; *A(x)* is the variable cross sectional area at *x;* *V(x, t)* is the beam lateral displacement; *f(x, t)* shear force. *K(x)* and *G(x)* are the variable foundation stiffness and foundation modulus per unit length of the elastic structure.

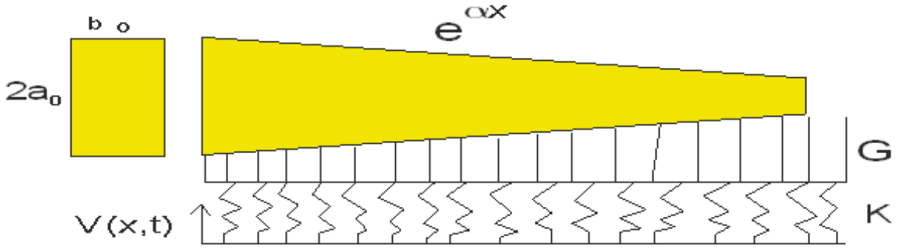


**Fig. 1. Geometry of the exponential beam on a bi-parametric foundation**

For a constant E and a variable cross section with respect to the horizontal axis (x), as illustrated in Fig.1, we can expand eq. (2.1) to obtain:

$$EI(x)\frac{\partial^4 V}{\partial x^4}+2E\frac{\partial I(x)}{\partial x}\frac{\partial^3 V}{\partial x^3}+E\frac{\partial^2 I(x)}{\partial x^2}\frac{\partial^2 V}{\partial x^2}+\rho A(x)\frac{\partial^2 V}{\partial t^2}+K(x)\,V(x,t)-G(x)\frac{\partial^2}{\partial x^2}\ V(x,t)=f(x,t) \qquad (2.2)$$

Denoting the beam's breadth and depth by $b = b_o$ and $a = \ 2a_o e^{\alpha x}$, respectively, one can easily write:

$$A(x)=ab=\ 2a_o b_o e^{\alpha x}, I(x)=\frac{1}{12}a^3 b=\ \frac{2}{3}a_o^3 b_o e^{3\alpha x} \qquad (2.3)$$

Where is a factor to show exponential rate of the beam. Substituting Eqn. (2.3) into (2.2), the following relation is obtained

$$\frac{2}{3}a_o^3 b_o e^{3\alpha x}\ \frac{\partial^4 V}{\partial x^4}+4E\alpha a_o^3 b_o e^{3\alpha x}\frac{\partial^2 V}{\partial x^2}+6Ea_o^3 b_o e^{3\alpha x}\frac{\partial^2 V}{\partial x^2}+2\rho a_o b_o e^{\alpha x}\frac{\partial^2 V}{\partial t^2}$$
$$+\ K(x)\,V(x,t)-\ G(x)\frac{\partial^2}{\partial x^2}\ V(x,t)\ =\ f(x,t) \qquad (2.4)$$

In this paper, the non- uniform elastic foundation K(x) and G(x) are given as

$$\mathrm{K}\,(\mathrm{x})=K_o(4x-3x^2+x^3); \qquad \mathrm{G}\,(\mathrm{x})=G_o(12-13x+6x^2-x^3) \qquad (2.5)$$

## 2.1 The boundary conditions

The boundary conditions depend on the constraints at the beam ends, however for a simply supported beam whose length is L, the vertical displacement at the beam ends are given as:

$$V(0,t)=V(L,t)=0 \qquad (2.6)$$

$$V_{xx}\left(0,t\right) = V_{xx}\left(L,t\right) = 0 \tag{2.7}$$

It is assumed that the initial conditions are

$$V\left(x,0\right) \;=\; 0 \;=\; V_{tt}\left(x,0\right) \tag{2.8}$$

Furthermore, the distributed load *f(x, t)* takes the form

$$f\left(x,t\right) = MgH\left(x-ct\right)\left[1-\frac{1}{g}\left(\frac{\partial^2}{\partial t^2}+2\mathrm{c}\frac{\partial^2}{\partial x\partial t}+c^2\frac{\partial^2}{\partial x^2}\right)\mathrm{V}\left(x,t\right)\right] \tag{2.9}$$

Where M is the mass of the moving load, g is the acceleration due to gravity and c is the velocity of the distributed mass, the time t is limited to that interval of time within which the mass is on the beam, that is

$$0 \le ct \le L \tag{2.10}$$

And $H\left(x-ct\right)$is the Heaviside function defined as

$$H\left(x-ct\right) = \begin{cases} 1, & x>0 \\ 0, & x<0 \end{cases} \tag{2.11}$$

With the properties,

$$(i) \qquad \frac{d}{dx}\left[H\left(x-ct\right)\right] = \delta\left(x-ct\right) \tag{2.12}$$

$$(ii) \qquad H\left(x-ct\right)f(x) = \begin{cases} 0, & x<ct \\ f(x), & x \ge ct \end{cases} \tag{2.13}$$

where $\delta\left(x-ct\right)$ represent the Dirac delta function and $H\left(x-ct\right)$ is a typical engineering function made to measure engineering applications which often involved functions that are either "on" or "off" .

## 2.2 Solution technique

In this section, in order to compute the response of the dynamic equation (2.4), we shall use an elegant technique called Galerkin's method often used in solving diverse problems involving mechanical vibrations Ojih (2013). This method requires that the solution of the deflection of the coupled equation is expressed as.

where $x_o$ is the equilibrium position of the longitudinal oscillating load, $\beta$is the longitudinal amplitude of oscillation of the load and $\alpha$ is the longitudinal frequency of the load.

$$V\left(x,t\right) = \sum_{m=1}^{N} y_m(t)\sin\frac{m\pi x}{L} \tag{2.14}$$

Thus, applying (2.14) to (2.4), one obtains

$$\begin{aligned}\sum_{m=1}^{N}\Bigg\{ & e^{\alpha x}sin\frac{m\pi x}{L}\ddot{y}_m(t) + \Big(\frac{m^4\pi^4Ea_o^2}{3\rho L^4}\Big)e^{3\alpha x}sin\frac{m\pi x}{L}y_m(t) - \Big(\frac{2m^3\pi^3Ea_o^2\alpha}{\rho L^3}\Big)e^{3\alpha x}cos\frac{m\pi x}{L}y_m(t) \\ & -\Big(\frac{3m^2\pi^2Ea_o^2\alpha^2}{\rho L^2}\Big)e^{3\alpha x}sin\frac{m\pi x}{L}y_m(t) + \Big(\frac{K(x)}{2\rho a_o b_o}\Big)sin\frac{m\pi x}{L}y_m(t) - \frac{m^2\pi^2G(x)}{2\rho a_o b_o L^2}sin\frac{m\pi x}{L}y_m(t) \\ +\frac{MH(x-ct)}{2\rho a_o b_o}&\left[sin\frac{m\pi x}{L}\ddot{y}_m(t) + 2c\left(\frac{m\pi}{L}\right)cos\frac{m\pi x}{L}\dot{y}_m(t) + c^2\left(\frac{m\pi}{L}\right)^2 sin\frac{m\pi x}{L}y_m(t)\right]\Bigg\} - \frac{Mg}{2\rho a_o b_o}H(x-ct) = 0\end{aligned} \tag{2.15}$$

To determine $y_m(t)$, the expressions on the left hand side of (2.15) are required to be orthogonal to the function $sin\frac{k\pi x}{L}$. Thus, equation (2.15) becomes

$$\int_0^L \left\{ \sum_{m=1}^{N} \left\{ e^{\alpha x} sin\frac{m\pi x}{L}\ddot{y}_m(t) + (\frac{m^4\pi^4 E a_o^2}{3\rho L^4})\mathrm{e}^{3\alpha x} sin\frac{m\pi x}{L} y_m(t) - (\frac{2m^3\pi^3 E a_o^2\alpha}{\rho L^3})\mathrm{e}^{3\alpha x} cos\frac{m\pi x}{L} y_m(t) \right.\right.$$
$$- (\frac{3m^2\pi^2 E a_o^2\alpha^2}{\rho L^2})\mathrm{e}^{3\alpha x} sin\frac{m\pi x}{L} y_m(t) + (\frac{K(x)}{2\rho a_o b_o}) sin\frac{m\pi x}{L} y_m(t) - \frac{m^2\pi^2 G(x)}{2\rho a_o b_o L^2} sin\frac{m\pi x}{L} y_m(t)$$
$$\left. + \frac{MH(x-ct)}{2\rho a_o b_o}\left[ sin\frac{m\pi x}{L}\ddot{y}_m(t) + 2c\left(\frac{m\pi}{L}\right) cos\frac{m\pi x}{L}\dot{y}_m(t) + c^2\left(\frac{m\pi}{L}\right)^2 sin\frac{m\pi x}{L} y_m(t)\right]\right\}$$
$$\left. - \frac{Mg}{2\rho a_o b_o} H(x-ct)\right\} sin\frac{k\pi x}{L}\ dx = 0 \tag{2.16}$$

Rearranging equation (2.16), yields

$$I_1\ddot{y}_m(t) + \Big(n_1 I_2 - n_2 I_3 - n_3 I_2 + n_4 I_4 + n_5 I_5\Big) y_m(t) + \frac{M}{2\rho a_o b_o}\Big[ I_6\ddot{y}_m(t) + 2cn_6 I_7 \dot{y}_m(t) + c^2 n_7 I_6 y_m(t)\Big] = P_o I_8 \tag{2.17}$$

Where

$$n_1 = \frac{m^4\pi^4 E a_o^2}{3\rho L^4},\ n_2 = \frac{2m^3\pi^3 E a_o^2\alpha}{\rho L^3},\ n_3 = \frac{3m^2\pi^2 E a_o^2\alpha^2}{\rho L^2},\ n_4 = \frac{K_o}{2\rho a_o b_o},\ n_5 = \frac{m^2\pi^2 G_o}{2\rho a_o b_o L^2}\ ,$$
$$n_6 = \frac{m\pi}{L},\ n_7 = \left(\frac{m\pi}{L}\right)^2,\ \ I_1 = \int_0^L \mathrm{e}^{\alpha x} sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx,\quad I_2 = \int_0^L \mathrm{e}^{3\alpha x} sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx,$$
$$I_3 = \int_0^L \mathrm{e}^{3\alpha x} cos\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx \qquad I_4 = \int_0^L (4x - 3x^2 + x^3) sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx,$$
$$I_5 = \int_0^L (12 - 13x + 6x^2 - x^3) sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx \qquad I_6 = \int_0^L H(x-ct)\, sin\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx,$$
$$I_7 = \int_0^L H(x-ct)\, cos\frac{m\pi x}{L} sin\frac{k\pi x}{L}\ dx,\qquad I_8 = \int_0^L H(x-ct)\, sin\frac{k\pi x}{L}\ dx,\qquad P_o = \frac{Mg}{2\rho a_o \boldsymbol{b_o}} \tag{2.18}$$

Further simplification and rearrangement of (2.15) after substituting the values of the integrals' results in (2.17) into it, one obtains

$$\ddot{y}_m(t) + {\beta_f}^2 y_m(t)$$
$$+\varepsilon_o\left\{\left[\frac{L^2}{8I_1} - \frac{4mL^2}{\pi^2 I_1}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\ \pi ct}{2n+1}\left(\frac{kr}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{r(r^2-4m^2)}\right)\right]\ddot{y}_m(t)\right.$$
$$+2c\left[-\frac{mkL}{2(m^2-k^2)I_1} + \frac{2mL}{\pi I_1}\sum_{n=0}^{\infty}\left(\frac{\sin(2n+1)\ \pi ct}{2n+1}\right)\left(\frac{k(r^2+m^2-k^2)}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{(r^2-4m^2)}\right)\right]\dot{y}_m(t)$$
$$\left.+c^2\left[\frac{-m^2\pi^2}{8I_1} - \frac{4m^3}{I_1}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\ \pi ct}{2n+1}\left(\frac{kr}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{r(r^2-4m^2)}\right)\right] y_m(t)\right\}$$
$$= \frac{P_o L}{k\pi \mathrm{I}_1}\left[\cos(\frac{k\pi ct}{L})\ - (-1)^k\right] \tag{2.19}$$

Where

$$\varepsilon_o = \frac{M}{2\rho a_o \boldsymbol{b_o L}} \tag{2.20}$$

$${\beta_f}^2 = \frac{n_1 I_2 - n_2 I_3 - n_3 I_2 + n_4 I_4 + n_5 I_5}{I_1} \tag{2.21}$$

Equation (2.2) is now the fundamental equation for the dynamic problem. It follows that two special cases of the equation (2.2) arise, namely the ***moving force*** and ***moving mass*** problems.

### 2.2.1 Non-prismatic bernoulli euler beam traversed by moving force

In this section, an approximate model of the differential equation describing the response of the dynamic problem is obtained by neglecting inertia terms, that is $\varepsilon_o = 0$. To this end, Eqn. (2.2) becomes

$$\ddot{y}_m(t) + {\beta_f}^2 y_m(t) = F_k\left[\cos(\theta t)\ - R_k\right] \tag{2.22}$$

Where

$$F_k = \frac{P_o L}{k\pi \mathrm{I}_1}, \qquad \theta = \frac{k\pi c}{L}, \qquad R_k = (-1)^k \tag{2.23}$$

Equation (2.22) is a second order ordinary differential equation, therefore, subjecting the equation to a Laplace transform defined as

$$\eta = \int_0^t (\sim)\, e^{-st} dt \tag{2.24}$$

In conjunction with the initial conditions defined in (2.8), gives the following algebraic equation

$$y_m(s) = F_k\left(\frac{s}{s^2+\theta^2} - \frac{R_k}{s}\right)\left(\frac{1}{s^2+{\beta_f}^2}\right) \tag{2.25}$$

Thus, the equation reduces to of finding Laplace inversion of (2.25), so that the Laplace inversion of $y_m(s)$ is convolution of (2.25) defined as

$$f(s) * g(s) = \int_0^t f(t-u)\, g(u)\, du \tag{2.26}$$

where

$$f(s) = \left(\frac{1}{s^2+{\beta_f}^2}\right) \quad and \quad g(s) = \left(\frac{s}{s^2+\theta^2} - \frac{R_k}{s}\right) \tag{2.27}$$

Therefore, Laplace inversion of (2.25) gives

$$y_m(t) = \frac{F_k}{\beta_f}\left(I_{1a} - R_k I_{1b}\right) \tag{2.28}$$

Where

$$I_{1a} = \int_0^t sin\beta_f\,(t-u) cos\theta u\ du \tag{2.29}$$

$$I_{1b} = \int_0^t sin\beta_f\,(t-u) du \tag{2.30}$$

It is easily to show that

$$I_{1a} = \frac{\beta_f}{{\beta_f}^2 - \theta^2}\left(cos\theta t - cos\beta_f t\right) \tag{2.31}$$

$$I_{1b} = \frac{1}{\beta_f}\left(1 - cos\beta_f t\right) \tag{2.32}$$

Substituting (2.31) and (2.32) into (2.28), after some rearrangement, one obtains

$$y_m(t) = \frac{F_k}{{\beta_f}^2\left({\beta_f}^2-\theta^2\right)}\left[{\beta_f}^2\left(cos\theta t - cos\beta_f t\right) - R_k\left({\beta_f}^2-\theta^2\right)\left(1-cos\beta_f t\right)\right] \tag{2.33}$$

When (2.33) is substituted to (2.14), one obtains

$$V(x,t) = \sum_{m=1}^{N}\frac{F_k}{{\beta_f}^2\left({\beta_f}^2-\theta^2\right)}\left[{\beta_f}^2\left(cos\theta t - cos\beta_f t\right) - R_k\left({\beta_f}^2-\theta^2\right)\left(1-cos\beta_f t\right)\right]\times\sin\frac{m\pi x}{L} \tag{2.34}$$

(2.34) is the transverse deflection of non-prismatic Bernoulli-Euler beam with exponentially varying thickness resting on variable elastic foundation under the action of moving distributed force.

### 2.2.2 Non-prismatic bernoulli euler beam traversed by moving mass

In this section, the solution to the entire equation (2.2) is sought when no terms of the coupled differential equation is neglected i.e. $\varepsilon_o \neq 0$ . Evidently an exact solution to these equations is not possible. Though the equations yield readily to numerical techniques, an analytical approximation method is desirable as a solutions so obtained often shed light on vital information about the vibrating system. Thus, we resort to a modification of the asymptotic method due to Strubble which is often used in treating oscillatory system. To this ends, equation (2.2) is rearranged to take the form

$$\ddot{y}_m(t) + \frac{2c\varepsilon_o Q_2}{1+\varepsilon_o Q_1}\dot{y}_m(t) + \frac{{\beta_f}^2 + c^2\varepsilon_o Q_3}{1+\varepsilon_o Q_1}y_m(t) = \frac{F_k}{1+\varepsilon_o Q_1}\left[\cos(\theta t)\ - R_k\right] \tag{2.35}$$

$$Q_1 = \frac{L^2}{8I_1} - \frac{4mL^2}{\pi^2 I_1}\sum_{n=0}^{\infty}\left(\frac{\cos(2n+1)\ \pi ct}{2n+1}\right)\left(\frac{kr}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{r\left(r^2-4m^2\right)}\right) \tag{2.36}$$

$$Q_2 = -\frac{mkL}{2\left(m^2-k^2\right)I_1} + \frac{2mL}{\pi I_1}\sum_{n=0}^{\infty}\left(\frac{\sin(2n+1)\ \pi ct}{2n+1}\right)\left(\frac{k\left(r^2+m^2-k^2\right)}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{\left(r^2-4m^2\right)}\right) \tag{2.37}$$

$$Q_3 = \frac{-m^2\pi^2}{8I_1} - \frac{4m^3}{I_1}\sum_{n=0}^{\infty}\left(\frac{\cos(2n+1)\ \pi ct}{2n+1}\right)\left(\frac{kr}{\left[r^2-(m+k)^2\right]\left[r^2-(m-k)^2\right]} + \frac{m}{r\left(r^2-4m^2\right)}\right) \tag{2.38}$$

We are interested in the modified frequency corresponding to the frequency of the free system due to the presence of the effects of rotatory inertia. An equivalent free system operator defined by the modified frequency then replaces equation (2.35).

To this end, we set the right hand side of (2.35) to zero and consider a parameter $\Gamma_o < 1$ for any arbitrary ratio $\varepsilon_o$, defined as

$$\Gamma_o = \frac{\varepsilon_o}{1+\varepsilon_o} \tag{2.39}$$

$$\varepsilon_o = \Gamma_o + 0\left({\varepsilon_o}^2\right) \tag{2.40}$$

Thus, in view of (2.40), we have

$$\ddot{y}_m(t) + \left(1+\Gamma_o Q_1\right)\left(2c\Gamma_o Q_2\right)\dot{y}_m(t) + \left(1+\Gamma_o Q_1\right)\left({\beta_f}^2 + c^2\Gamma_o Q_3\right)y_m(t) = 0 \tag{2.41}$$

It is observed that if we set $\Gamma_o = 0$, in (2.41),the result will be solution corresponding to the case in which the inertia effect of the mass of the system is regarded as negligible, then the solution of equation (2.41) becomes

$$y_m(t) = \Delta_a cos\left[\beta_f t - \Omega_f\right] \tag{2.42}$$

Where $\Delta_a$ and $\Omega_f$ are constants. Since $\Gamma_o < 1$, Strubble's technique required that the asymptotic solution of the homogeneous part of the equation (2.41) be of the form

$$y_m(t) = \ (\mathrm{m,t})\, cos\left[\beta_f t - \phi(m,t)\right] + \Gamma_o y_1(t) + 0\left({\Gamma_o}^2\right) \tag{2.43}$$

Where $\ (\mathrm{m,t})$ and $\phi(m,t)$ are slowly varying functions of time.

The variational equations [2,4] describing the behavior of $\ (\mathrm{m,t})$ and $\phi(m,t)$ during the motion of the force are obtained by substituting (2.43) into (2.41). Thus, we have

$$\begin{aligned} &\left[2\dot{\phi}(x,t)\beta_f - {\beta_f}^2\Gamma_o Q_1 + c^2\Gamma_o Q_3\right]\left[\ (\mathrm{m,t})\, cos\left[\beta_f t - \phi(m,t)\right]\right] \\ -&\left[2\dot{\psi}(x,t)\beta_f + 2c\Gamma_o Q_2\right]\left[\beta_f sin\left[\beta_f t - \phi(m,t)\right]\right] \quad + \Gamma_o y_1(t) = 0 \end{aligned} \tag{2.44}$$

Extracting those terms which contribute to the variational equation to $0(\Gamma_o)$, we have

$$\begin{aligned} &\left[2\dot{\phi}(x,t)\beta_f - \tfrac{{\beta_f}^2\Gamma_o L^2}{8I_1} + \tfrac{c^2\Gamma_o m^2\pi^2}{8I_1}\right]\left[\ (\mathrm{m,t})\, cos\left[\beta_f t - \phi(m,t)\right]\right] \\ -&\left[2\dot{\psi}(x,t)\beta_f + \tfrac{c\Gamma_o mkL}{\left(m^2-k^2\right)I_1}\right]\left[\beta_f sin\left[\beta_f t - \phi(m,t)\right]\right] \quad = 0 \end{aligned} \tag{2.45}$$

Setting the coefficients of $sin\left[\beta_f t - \phi(m,t)\right]$ and $cos\left[\beta_f t - \phi(m,t)\right]$ to zero, we have the

$$\dot{\psi}(x,t) = -\frac{c\Gamma_o mkL}{2\left(m^2-k^2\right)I_1} \tag{2.46}$$

and

$$\dot{\phi}(x,t) = \frac{\beta_f\Gamma_o L^2}{16I_1} - \frac{c^2\Gamma_o m^2\pi^2}{16I_1\beta_f} \tag{2.47}$$

respectively. Equations (2.46) and (2.47) which implies

$$\psi(x,t) = A_0 e^{\vartheta t} \tag{2.48}$$

and

$$\phi(x,t) = \frac{\beta_f\Gamma_o L^2}{16I_1}\left(1 + \frac{c^2m^2\pi^2}{\beta^2{}_f L^2}\right)t + C_m \tag{2.49}$$

are termed the variational equations.

Therefore, when the effect of the cross-sectional dimensions or rotatory inertia is considered, the first approximation to the homogeneous system (2.41) is

$$y_m(t) = \ (\mathrm{m,t})\, cos\left[\beta_m t - \phi(m,t)\right] \tag{2.50}$$

where

$$\beta_m = \beta_f\left[1 - \frac{\Gamma_o L^2}{16I_1}\left(1 + \frac{c^2m^2\pi^2}{{\beta_f}^2 L^2}\right)\right] \tag{2.51}$$

is the modified frequency due to the effect of cross-sectional dimensions of the beam. It is observed that when $\Gamma_o = 0$, we recover the frequency of the moving force problem when the inertia influence is neglected.

Thus, equation (2.41) takes the form

$$\ddot{y}_m(t) + {\beta_m}^2 y_m(t) = 0 \tag{2.52}$$

and equation (2.35) then becomes

$$\ddot{y}_m(t) + {\beta_m}^2 y_m(t) = F_k\left[\cos(\theta t)\ - R_k\right] \tag{2.53}$$

Since equation (2.53) is analogous to equation (2.22), therefore, the solution to (2.53) after inversion is

$$V(x,t) = \sum_{m=1}^{N} \frac{F_k}{{\beta_m}^2\left({\beta_m}^2 - \theta^2\right)}\left[{\beta_m}^2\left(cos\theta t - cos\beta_m t\right) - R_k\left({\beta_m}^2 - \theta^2\right)\left(1 - cos\beta_m t\right)\right] \times \sin\frac{m\pi x}{L} \tag{2.54}$$

Equation (2.54) represent the corresponding dynamic response to a moving mass of our non-prismatic Bernoulli-Euler beam with exponentially varying thickness resting on variable elastic foundation.

# 3 Results and Discussion

The transverse displacement of a non-prismatic Bernoulli-Euler beam may grow without bound. Evidently, from equation (2.34) that the simply supported beam traversed by a moving force will be in a state of resonance when

$$\beta_f = \frac{k\pi c}{L} \tag{3.1}$$

While equation (2.54) shows that the same beam traversed by a moving mass encounter a resonance effect at

$$\beta_m = \frac{k\pi c}{L} \tag{3.2}$$

Consequently,

$$\beta_f = \frac{\frac{k\pi c}{L}}{\left[1 - \frac{\Gamma_o L^2}{16 I_1}\left(1 + \frac{c^2 m^2 \pi^2}{{\beta_f}^2 L^2}\right)\right]} \tag{3.3}$$

Thus, from equations (3.1) and (3.3), for the same natural frequency, the critical speed for the system made up of a simply supported non-prismatic Bernoulli-Euler beam with exponentially varying thickness traversed by a moving force is greater than that under the influence of a moving mass. Thus, resonance is attained earlier in the moving distributed mass system than in the moving distributed force system. For the purpose of numerical analysis of the forgoing problem, the velocity of the moving load and the length of non-prismatic beam are 30m/s and 15m respectively. Furthermore, E = 2.02e+11, $a_o$= 0 .25, $b_o$=10, = -0.12. The deflection profile of a non-prismatic Bernoulli-Euler beam with exponentially varying thickness traversed by a moving force is shown in fig. (2a) - fig. (2d). It is observed that as the values of the varying parameters are increased, the response amplitudes decreased. The same effect is shown for the moving mass model which are shown in fig. (2e) - fig. (2h). Fig. (2i) illustrates the response of the beam for moving force and moving mass. Clearly, the response amplitude due to the moving mass is greater than that due to moving force. Thus, the moving force solution is not always an upper bound for the accurate solution for the beam problem.

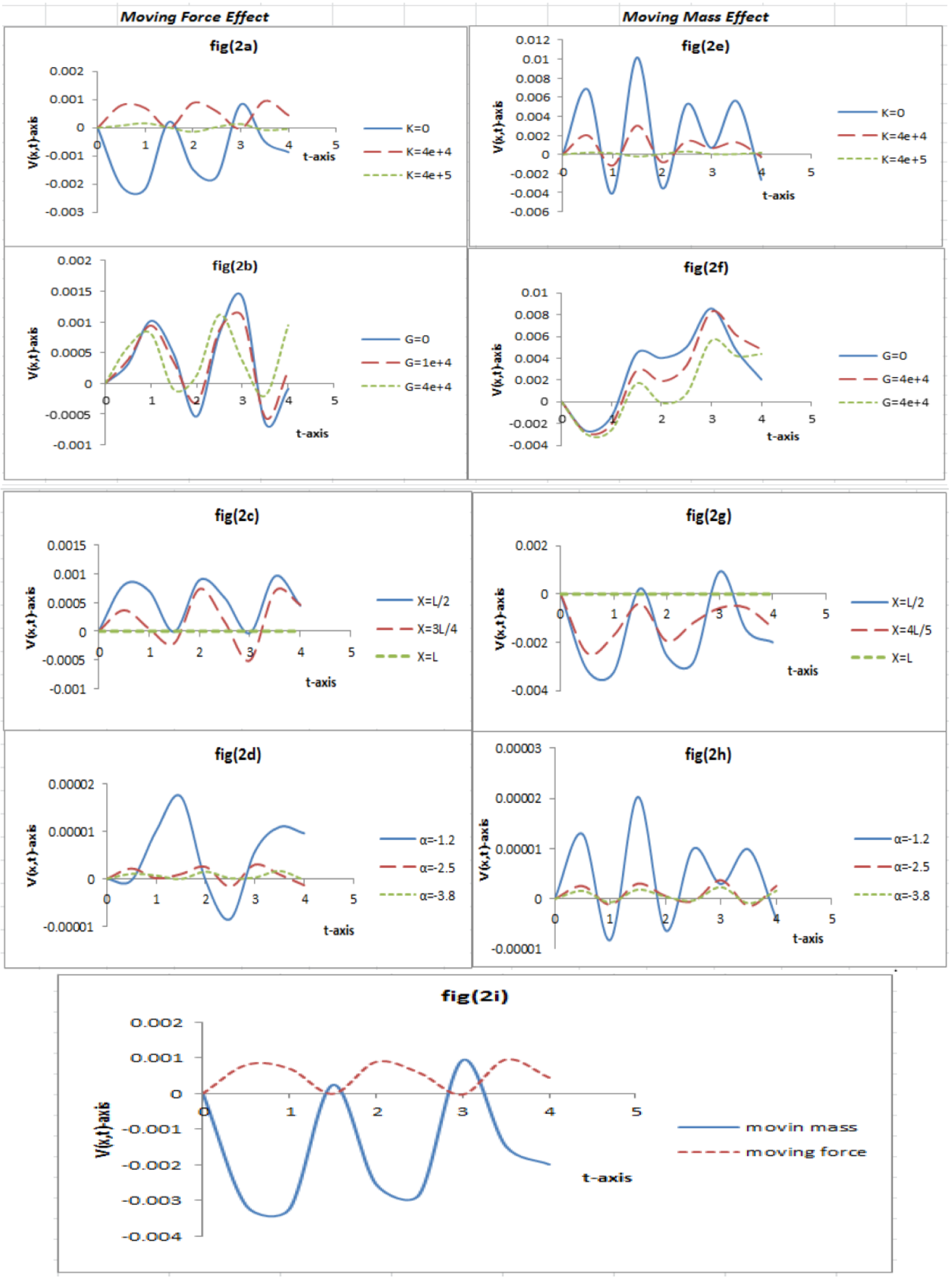


**Fig. 2. The deflection profile of a non-prismatic Bernoulli-Euler beam with exponentially varying thickness traversed by a (i)moving force fig(2a-2d) (ii) moving mass fig(2e-2h) and (iii) moving force and moving mass comparison**

# 4 Conclusion

An analytical solution is presented for the deflection response of non-prismatic Bernoulli-Euler beam under the action of a distributed load moving with constant velocity. The solution technique is based on Galerkin's method and modification of the asymptotic method. The analysis exhibited the following features:

1. The critical speeds of the system increases with an increase in the values of foundation stiffness, foundation modulus and exponential factor in the problem of non-prismatic Bernoulli-Euler beam with exponentially thickness resting on variable bi-parametric foundation.
2. As the foundation parameters increased, the transverse deflection of the beam model decreased.

Thus, the risk of resonance was reduced as the value of the foundation parameters increased.

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## Competing Interests

Authors have declared that no competing interests exist.

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