Dualities between "Kamal & Mahgoub Integral Transforms" and "Some Famous Integral Transforms"

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Authors' contributions

This work was carried out in collaboration between all authors. All authors read and approved the final manuscript.

Original Research Article

ABSTRACT

Abdelilah Kamal and Mohand Mahgoub recently introduced new integral transforms separately by the names the "Kamal transform" and "Mahgoub transform" to facilitate the solution of differential and integral equations. However, in this paper, these newly introduced integral transforms will closely be studied in relation to the some existing famous integral transforms defined in the time domain. The study will also try to establish the duality relations existing between these new integral transforms and in particular, the Laplace, Sumudu, Elzaki and Aboodh integral transforms. Further, supporting illustrations obtained from some test functions as examples are will be presented.

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1. INTRODUCTION

Integral transforms are mathematical tools for solving differential and integral equations for centuries. However, this old area has recently got a center stage among many researchers by introducing many integral transforms among which are [1-6]. Furthermore, these transforms are studied and analyzed in order to establish relationships and connections existing among them among which are [7-9]. However, in this paper, the recently introduced integral transforms by Abdelilah Kamal [10] and Mohand Mahgoub [11] are closely studied in relation to the some existing famous integral transforms that are defined in the time domain. The study also tried to establish the duality relations existing between these newly introduced integral transforms and Laplace, Sumudu, Elzaki and Aboodh integral transforms as particular cases and further some test functions are examined as examples. Further, applications of these new integral transforms to the solution of partial differential equations are demonstrated in [12] and [13], respectively for interested readers.

The paper is organized as follows: In Section 2, we present the basic concept of Kamal and Mahgoub transforms differently. Section 3 presents Kamal transform dualities, while Section 4 presents Mahgoub transform dualities and Section 5 gives the conclusion.

2. KAMAL AND MAHGOUB TRANSFORMS

In this section, we present the basic concept of Kamal and Mahgoub transforms.

2.1 Kamal Transform

The Kamal transform is a newly introduced integral transform similar to Laplace transform and other integral transforms [1-6] that are defined in the time domain $t \geq 0$, and for functions of exponential order. We consider functions in the set B defined by:

$$B = \{ f : |f(t)| < Q \, e^{\frac{|t|}{k_j}} \text{if } t \in (-1)^j \times [0, \infty), \\ j = 1,2; \, (Q, k_1, k_2 > 0) \}.$$

For a given function in the set B, the constant Q must be finite number; k_1, k_2 may be finite or infinite.

Then, the Kamal transform denoted by the operator $K(.)$ for function of exponential order and belonging to set B is defined (by [10]) by the integral equation:

$$G(u) = K[f(t)] = \int_0^\infty e^{\frac{-t}{u}} f(t) \, dt \, , u \in (k_1, k_2) \tag{1}$$

2.1.1 Kamal transform for derivatives

For any given function $f(t)$ defined over the set B, Kamal transform of derivatives are given as follows:

$$K[f'(t)] = u^{-1}G(u) - f(0),$$

$$K[f''(t)] = u^{-2}G(u) - u^{-1}f(0) - f'(0),$$

$$K[f^{(n)}(t)] = u^{(-n)}G(u) - \sum_{k=0}^{n-1} u^{k-n+1}f^{(k)}(0),$$
$$\text{for } n = 0,1,2, \ldots$$

2.2 Mahgoub Transforms

The newly introduced integral transform by Mahgoub [11] is an integral transform that has the same setting with the Kamal integral transform. Thus, the Mahgoub integral transform denoted by the operator $M(.)$ is defined by:

$$H(u) = M[f(t)] = u \int_0^\infty e^{-ut} f(t) \, dt \, , u \in (k_1, k_2) \tag{2}$$

2.2.1 Mahgoub transform for derivatives

For any given function $f(t)$ defined over the set B , Mahgoub transform of derivatives are given as follows:

$$M[f'(t)] = uH(u) - u \, f(0),$$

$$M[f''(t)] = u^2H(u) - u \, f'(0) - u^2 \, f(0),$$

$$M[f^{(n)}(t)] = u^{(n)}H(u) - \sum_{k=0}^{n-1} u^{n-k}f^{(k)}(0),$$
$$\text{for } n = 0,1,2, \ldots$$

2.3 Connection Between Kamal and Mahgoub Transforms

To examine the connection between Kamal and Mahgoub transforms, we give the following theorem:

Theorem 2.3.1

Let $f(t) \in B$ with Kamal transform $G(u)$ and Mahgoub transform $H(u)$, then

$$G(u) = u\, H\left(\frac{1}{u}\right) \qquad (3)$$

and

$$H(u) = u\, G\left(\frac{1}{u}\right). \qquad (4)$$

Proof: To prove Eq. (3), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)\, dt = u\left(\frac{1}{u} \int_0^\infty e^{\frac{-t}{u}} f(t)\, dt\right)$$

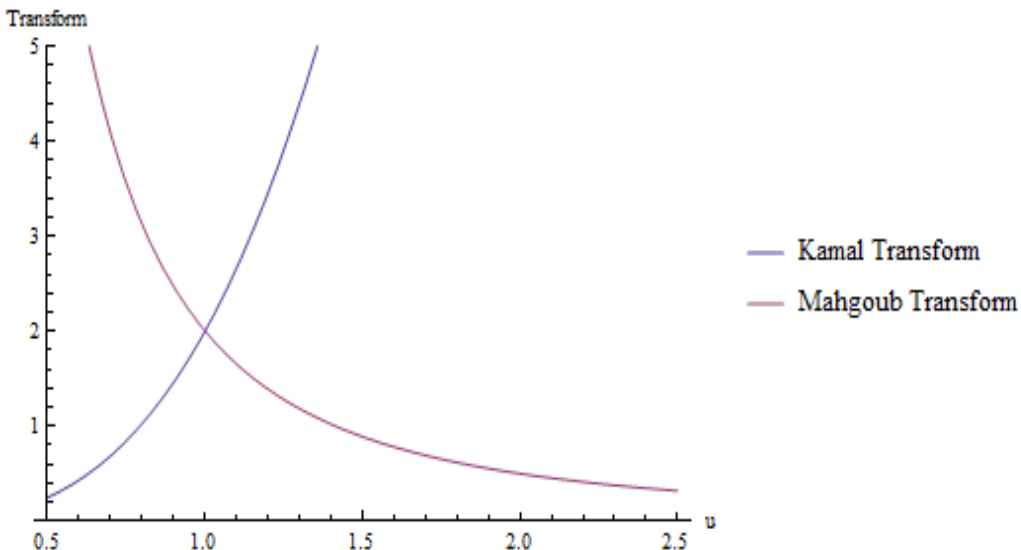Thus, from Eq. (2)

$$G(u) = u\, H\left(\frac{1}{u}\right).$$

To prove Eq. (4), from the Eq. (2), we have

$$H(u) = u \int_0^\infty e^{-ut} f(t)\, dt = u\left(\int_0^\infty e^{\frac{-t}{\left(\frac{1}{u}\right)}} f(t)\, dt\right).$$

Thus, from Eq. (1), we get $H(u) = uG\left(\frac{1}{u}\right)$.

Example 2.3.2

Consider the function $f(t) = t^2$. Kamal and Mahgoub transforms are obtained respectively as follows: $G(u) = 2u^3$ and $H(u) = \frac{2}{u^2}$, and their graph is given in Fig. 1.

3. KAMAL TRANSFORM DUALITIES

In this section, we present the Kamal transform dualities starting with the Laplace, Sumudu, Elzaki and Aboodh transforms.

3.1 Kamal -Laplace Duality

Laplace transform of $f(t)$ for $t \geq 0$, denoted by $F(u)$, is defined by the equation

$$F(u) = L[f(t)] = \int_0^\infty e^{-ut} f(t)\, dt \qquad (5)$$

Fig. 1. Plots for Kamal and Mahgoub transforms of $f(t) = t^2$

Theorem 3.1.1:

Let $f(t) \in B$ with Kamal transform $G(u)$ and Laplace transform $F(u)$, then

$$G(u) = F\left(\frac{1}{u}\right) \qquad (6)$$

and

$$F(u) = G\left(\frac{1}{u}\right) \qquad (7)$$

Proof: To prove Eq. (6), let $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)\, dt = \int_0^\infty e^{-\left(\frac{1}{u}\right)t} f(t)\, dt.$$

Thus, from Eq. (5) $G(u) = F\left(\frac{1}{u}\right)$.

To prove Eq. (7), from the Eq. (5), we have

$$F(u) = \int_0^\infty e^{-ut} f(t)\, dt = \int_0^\infty e^{\frac{-t}{\left(\frac{1}{u}\right)}} f(t)\, dt.$$

Thus, from Eq. (1), we get $F(u) = G\left(\frac{1}{u}\right)$.

3.2 Kamal-Sumudu Duality

Sumudu transform of $f(t)$ for $t \geq 0$ denoted by $N(u)$, is defined by the equation

$$N(u) = S[f(t)] = \int_0^\infty e^{-t} f(ut)\, dt, \ k_1, k_2 > 0. \ (8)$$

Theorem 3.2.1:

Let $f(t) \in B$ with Kamal transform $G(u)$ and Sumudu transform $N(u)$, then

$$G(u) = uN(u) \qquad (9)$$

and

$$N(u) = \frac{1}{u} G(u) \qquad (10)$$

Proof: To prove Eq. (9), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)\, dt.$$

Now let, $w = \frac{t}{u} \Rightarrow t = uw$ and $dt = udw$. Then,

$$G(u) = \int_0^\infty e^{-w} f(uw)\, udw$$

$$= u\left(\int_0^\infty e^{-w} f(uw)\, dw\right).$$

Hence, on comparing with Eq. (8), we thus obtain

$$G(u) = uN(u).$$

To prove Eq. (10), from Eq. (8),

$$N(u) = \int_0^\infty e^{-t} f(ut)\, dt.$$

Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$N(u) = \int_0^\infty e^{\frac{-w}{u}} f(w) \frac{dw}{u} = \frac{1}{u}\left(\int_0^\infty e^{\frac{-w}{u}} f(w)\, dw\right).$$

Hence, on comparing with Eq. (1), we thus obtain

$$N(u) = \frac{1}{u} G(u).$$

3.3 Kamal-Elzaki Duality

Elzaki transform of $f(t)$ for $t \geq 0$, denoted by $T(u)$, is defined by the equation

$$T(u) = E[f(t)] = u^2 \int_0^\infty e^{-t} f(ut)\, dt, \ u \in (k_1, k_2). \qquad (11)$$

Theorem 3.3.1:

Let $f(t) \in B$ with Kamal transform $G(u)$ and Elzaki transform $T(u)$, then

$$G(u) = \frac{1}{u} T(u) \qquad (12)$$

and

$$T(u) = uG(u) \qquad (13)$$

Proof: To prove Eq. (12), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)\, dt.$$

Let, $w = \frac{t}{u} \Rightarrow t = uw$ and $dt = udw$. Then,

$$G(u) = \int_0^\infty e^{-w} f(uw)\, udw = u\left(\int_0^\infty e^{-w} f(uw)\, dw\right) = \frac{1}{u}\left(u^2 \int_0^\infty e^{-w} f(uw)\, dw\right)$$

Hence, on comparing with Eq. (11), we thus obtain

$$G(u) = \frac{1}{u}T(u).$$

To prove Eq. (13), from Eq. (11),

$$T(u) = u^2 \int_0^\infty e^{-t} f(ut)dt.$$

Let, $t = \frac{w}{u} \Rightarrow dt = \frac{dw}{u}$. Then,

$$T(u) = u^2 \int_0^\infty e^{\frac{-w}{u}} f(w)\frac{dw}{u}$$
$$= u\left(\int_0^\infty e^{\frac{-w}{u}} f(w)\,dw\right).$$

Hence, on comparing with Eq. (1), we thus obtain

$$T(u) = uG(u).$$

3.4 Kamal-Aboodh Duality

Aboodh transform of $f(t)$ for $t \geq 0$, denoted by $K(u)$, is defined by the equation

$$K(u) = A[f(t)] = \frac{1}{u}\int_0^\infty e^{-ut} f(t)dt, \ u \in (k_1, k_2) \tag{14}$$

Theorem 3.4.1:

Let $f(t) \in B$ with Kamal transform $G(u)$ and Aboodh transform $K(u)$, then

$$G(u) = \frac{1}{u}K\left(\frac{1}{u}\right) \tag{15}$$

And

$$K(u) = \frac{1}{u}G\left(\frac{1}{u}\right) \tag{16}$$

Proof: To prove Eq. (15), hence $f(t) \in B$, then from Eq. (1)

$$G(u) = \int_0^\infty e^{\frac{-t}{u}} f(t)\,dt = \frac{1}{u}\left(\frac{1}{\frac{1}{u}}\int_0^\infty e^{-\left(\frac{1}{u}\right)t} f(t)\,dt\right).$$

Thus, from Eq. (14)

$$G(u) = \frac{1}{u}K\left(\frac{1}{u}\right).$$

To prove Eq. (16), from the Eq. (14), we have

$$K(u) = \frac{1}{u}\int_0^\infty e^{-ut} f(t)dt = \frac{1}{u}\left(\int_0^\infty e^{\frac{-t}{\left(\frac{1}{u}\right)}} f(t)dt\right)$$

Thus, from Eq. (1), we get:

$$K(u) = \frac{1}{u}G\left(\frac{1}{u}\right).$$

Example 3.5

Consider the function $f(t) = e^{-2t}$. We obtain the following transforms:

$$F(u) = \frac{1}{u+2}, N(u) = \frac{1}{1+2u}, T(u) = \frac{u^2}{1+2u}, K(u) = \frac{1}{u^2+2u} \text{ and } G(u) = \frac{u}{1+2u}$$

Fig. 2 gives the illustration of the above transforms of the function under consideration.

4. MAHGOUB TRANSFORM DUALITIES

In the same pattern, we present Mahgoub transform dualities, at the same time maintaining the definitions of the integral transforms presented in section 3.

4.1 Mahgoub-Laplace Duality

Theorem 4.1.1:

Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Laplace transform $F(u)$, then

$$H(u) = uF(u) \tag{17}$$

and

$$F(u) = \frac{1}{u}H(u) \tag{18}$$

Proof: To prove Eq. (17), hence $f(t) \in B$, then from Eq. (2)

$$H(u) = u\int_0^\infty e^{-ut} f(t)\,dt = u\left(\int_0^\infty e^{-ut} f(t)\,dt\right)$$

Thus, from Eq. (5):

$$H(u) = uF(u)$$

To prove Eq. (18), from the Eq. (5), we have

$$F(u) = \int_0^\infty e^{-ut} f(t)\,dt = \frac{1}{u}\left(u\int_0^\infty e^{-ut} f(t)\,dt\right)$$

Thus, from Eq. (2), we get

$$F(u) = \frac{1}{u} H(u)$$

4.2 Mahgoub-SumuduDuality

Theorem 4.2.1:

Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Sumudu transform $N(u)$, then

$$H(u) = N\left(\frac{1}{u}\right) \tag{19}$$

and

$$N(u) = H\left(\frac{1}{u}\right) \tag{20}$$

Proof: To prove Eq. (19), let $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t)\, dt.$$

Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$ .Then,

$$H(u) = \int_0^\infty e^{-w} f\left(\frac{w}{u}\right) dw = \int_0^\infty e^{-w} f\left(\frac{1}{u}w\right) dw.$$

Hence, on comparing with Eq. (8), we thus obtain:

$$H(u) = N\left(\frac{1}{u}\right)$$

To prove Eq. (20), from Eq. (8),

$$N(u) = \int_0^\infty e^{-t} f(ut)\, dt.$$

Let,$= ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$N(u) = \int_0^\infty e^{\frac{-w}{u}} f(w) \frac{dw}{u}$$

$$= \frac{1}{u}\left(\int_0^\infty e^{-\left(\frac{1}{u}\right)w} f(w)\, dw \right).$$

Hence, on comparing with Eq. (2), we thus obtain:

$$N(u) = H\left(\frac{1}{u}\right)$$

4.3 Mahgoub-Elzaki Duality

Theorem 4.3.1:

Let $f(t) \in B$ with Mahgoub transform $H(u)$, and Elzaki transform $T(u)$, then

$$H(u) = u^2 T(u) \tag{21}$$

And

$$T(u) = u^2 H\left(\frac{1}{u}\right) \tag{22}$$

Proof: To prove Eq. (21), hence$f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t)\, dt.$$

Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$.Then,

$$H(u) = \int_0^\infty e^{-w} f\left(\frac{w}{u}\right) dw$$

$$= u^2 \left(\left(\frac{1}{u}\right)^2 \int_0^\infty e^{-w} f\left(\frac{1}{u}w\right) dw \right).$$
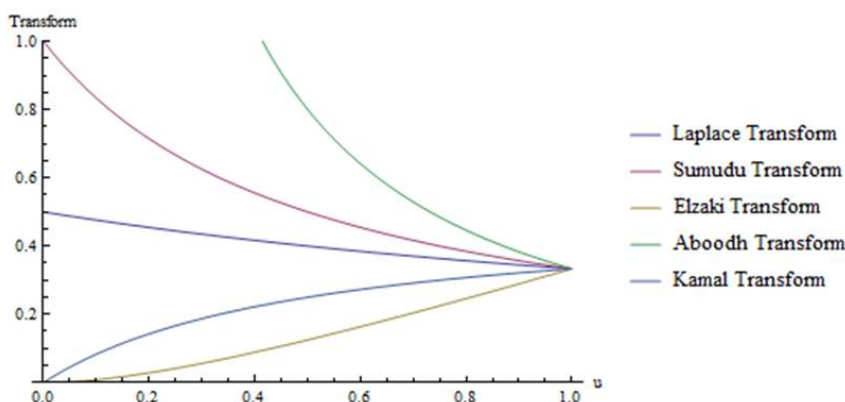

Fig. 2. Plots for Laplace, Sumudu, Elzaki, Aboodh and Kamal transforms of $f(t) = e^{-2t}$

Hence, on comparing with Eq. (11), we thus obtain:

$$H(u) = u^2 T\left(\frac{1}{u}\right)$$

To prove Eq. (22), from Eq. (11)

$$T(u) = u^2 \int_0^\infty e^{-t} f(ut)\, dt.$$

Let, $w = ut \Rightarrow t = \frac{w}{u}$ and $dt = \frac{dw}{u}$. Then,

$$T(u) = u \int_0^\infty e^{\frac{-w}{u}} f(w)\, dw$$
$$= u^2 \left(\frac{1}{u} \int_0^\infty e^{-\left(\frac{1}{u}\right)w} f(w)\, dw\right)$$

Hence, on comparing with Eq. (2), we thus obtain

$$T(u) = u^2 H\left(\frac{1}{u}\right)$$

4.4 Mahgoub-Aboodh Duality

Theorem 4.4.1:

Let $f(t) \in B$ with Mahgoub transform $H(u)$ and Aboodh transform $K(u)$, then

$$H(u) = u^2 K(u) \tag{23}$$

and

$$K(u) = \frac{1}{u^2} H(u) \tag{24}$$

Proof: To prove Eq. (23), hence $f(t) \in B$, then from Eq. (2)

$$H(u) = u \int_0^\infty e^{-ut} f(t)\, dt = u^2 \left(\frac{1}{u} \int_0^\infty e^{-ut} f(t)\, dt\right)$$

And, we obtain from Eq. (14):

$$H(u) = u^2 K(u)$$

To prove Eq. (24), from the Eq. (14), we have

$$K(u) = \frac{1}{u} \int_0^\infty e^{-ut} f(t)\, dt = \frac{1}{u^2} \left(u \int_0^\infty e^{-ut} f(t)\, dt\right)$$

Thus, from Eq. (2), we get

$$K(u) = \frac{1}{u^2} H(u)$$

Example 4.5

Consider the function $f(t) = t^2$. We obtain the following transforms:

$$F(u) = \frac{2}{u^3}, N(u) = 2u^2,\ T(u) = 2u^4,\ K(u) = \frac{2}{u^4}$$

and

$$H(u) = \frac{2}{u^2}.$$

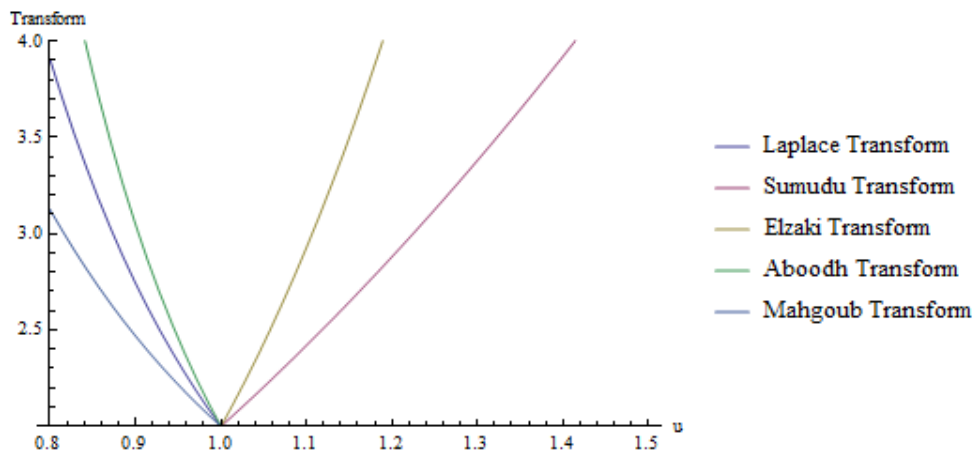Fig. 3 gives the illustrations of the connections between the transforms under consideration as follows:

Fig. 3. Plots for Laplace, Sumudu, Elzaki, Aboodh and Mahgoub transforms of $f(t) = t^2$

7

5. CONCLUSION

In conclusion, the duality relations existing between the Kamal and Mahgoub integral transforms with the Laplace, Sumudu, Elzaki and Aboodh integral transforms are established together with their converses. The dualities are presented in forms of theorems and are later proved in a friendly manner. Some test functions are considered as examples and illustrated graphically with the help of Mathematica software. Thus, we therefore recommend the use of these new integral transforms in solving various differential and integral equations that are found in mathematical physics and engineering problems; having established their duality relations with some famous integral transforms.

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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