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Modeling Exponential Growth and Exponential Decay Real Phenomena by ψ-Caputo Fractional Derivative

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Authors' contributions

This work was carried out in collaboration between both authors. They equally contributed in developing the theory and the computation. Both authors read and approved the final manuscript.

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Abstract

The concept of 'ψ-Caputo' fractional derivative is discussed in this article. This method is based on the fractional derivative in Caputo sense of a function with respect to another function ψ, called kernel. The kernel function ψ, is any increasing function such that $\psi'(x) \neq 0,\ \forall x \in [a,b]$. Experimental studies are used to support the fact that fractional approach of solving differential equations is often better than the classical ordinary approach. The solution to two exponential decay models and one exponential growth model are built using the classical approach and the kernel approach. Several kernel functions are considered and their performances evaluated.

Keywords: Exponential decay; exponential growth; ψ-Caputo fractional derivative; optimization; initial value problems.

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1 Introduction

Fractional calculus was discovered more than three centuries ago. Earlier works on Fractional calculus focused only on theories. Hence, existence and uniqueness of solution to problem were studied, without further materialization. [1-5] are some known references where the theory of existence and the uniqueness of fractional differential equations and their solutions are found.

The classical approach involving system of integer-order of differentiation may sometime provide solution with high magnitude error term. However, fractional order of differential equation has been proven efficient, and may produce lower magnitude error term in many situations where the classical approaches fail to do so. The efficiency of the fractional differentiation approach has been proven in various fields of sciences such as physics, chemistry, epidemiology, finance and biology sciences [6-12], these are few of them just to mention.

It is sometime difficult to find the analytical solution of some differential equations. Difficulty arises usually in nonlinear cases. To overcome such difficulty, numerical approaches are recommended. The solution of an equation using numerical approach is obtained through iterations. In [13-18], the authors discussed some numerical approaches to solve fractional differential equations and to optimize the fractional order of derivative.

In this work, we added our contribution to what have been done so far by many researchers such as Almeida [19] who introduced fractional derivative of a function with respect to another function, called the ψ -Caputo. Almeida et al [12,19] have proven further that the fractional order of differential equation is better than the classical in modeling the population growth, a country's GDP, the Newtown law of cooling and many others real world phenomena. We proved that, the ψ -Caputo fractional differential equation is better than their integer-order counterpart for modeling the radioactivity decay and the RC circuit charging and discharging processes, provided that a suitable kernel ψ is selected [20].

2 Preliminaries

Definition 2.1 [20] The Riemann-Liouville fractional integral of order $\alpha > 0$ for a function $g : [0, +\infty] \to \mathrm{R}$ is defined as

$$\left({}_{RL} I_{0^+}^{\alpha} g \right)(\tau) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} g(s)\, ds.$$

Provided that the right hand side of the integral is pointwise defined on $(0, +\infty)$ and Γ is the gamma function $\Gamma(\upsilon) = \int_0^{\infty} e^{-t} t^{\upsilon-1} dt, \forall \upsilon > 0.$

Definition 2.2 [20] The Caputo derivative of order $\alpha > 0$ for a function $g : [0, +\infty] \to \mathrm{R}$ is defined as

$$\left({}_{C} D_{0^+}^{\alpha} g \right)(t) = \begin{cases} \int_0^t \dfrac{(t-s)^{n-\alpha-1} g^{(n)}(s)}{\Gamma(n-\alpha)} ds & , n-1 < \alpha < n,\ \alpha \in R, \\ g^{(n)}(t) & , \alpha \in N. \end{cases}$$

Where $n=\lceil\alpha\rceil+1$, $\lceil\alpha\rceil$ is the integer part of α.

Definition 2.3 [19] Let $\alpha>0$, $g\in L^1[a,b]$ and $\psi\in C^1[a,b]$ be an increasing function with $\psi'(x)\neq 0,\ \forall x\in[a,b]$ then $I_{0^+}^{\alpha,\psi}g(t)$ denotes the fractional integral of g w.r.t ψ and it is given by

$$I_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)\big(\psi(t)-\psi(s)\big)^{\alpha-1}g(s)ds.$$

Definition 2.4 [19] Let $\alpha>0$, $g,\psi\in C^n[a,b]$ with ψ be an increasing function and $\psi'(x)\neq 0,\ \forall x\in[a,b]$ then ${}_C D_{0^+}^{\alpha,\psi}g(t)$ denotes the fractional derivative of g w.r.t ψ and it is given by

$${}_C D_{0^+}^{\alpha,\psi}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t\psi'(s)\big(\psi(t)-\psi(s)\big)^{n-\alpha-1}\left(\frac{1}{\psi'(s)}\frac{d}{dt}\right)^n g(s)ds.$$

Lemma 2.1 [19] Let $\alpha>0$, n a natural number such that $\alpha\in(n-1,n)$ if $g,\psi\in C^n[a,b]$ then

$$I_{0^+}^{\alpha,\psi}\left({}_C D_{0^+}^{\alpha,\psi}g\right)(t)=g(t)-\sum_{i=0}^{n-1}\frac{\left(\frac{1}{\psi'(s)}\frac{d}{dt}\right)^i g(0)}{i!}\big(\psi(t)-\psi(0)\big)^i.$$

3 Methodology

To investigate the power of the ψ-Caputo fractional differential equation over the classical differential equation, we considered some linear model differential equations whose classical solutions are given either in the form of exponential decay or exponential growth curve model. For each model, the true data obtained from experimental study is considered, and then the model based on classical approach of differential equation is defined alongside its counterpart derived from the ψ-Caputo fractional differential equation. Both models are used to fit the original data set. The percentage of the root square deviation error is used to compare the performance of each of the methods applied.

Consider an experimental data vector of size *n*, obtained from a time dependent process such that the i^{th} value is denoted by $y_i=y(t_i)$. If the original data is predicted by a function $\hat{y}$, in a way that the i^{th} fitted data value is denoted by $\hat{y}=\hat{y}(t_i)$, the root-mean-squared deviation is computed (see [21]) by

$$RMSD=\sqrt{\frac{\sum_{t=0}^{n}\big(y(t)-\hat{y}(t)\big)^2}{n}}. \tag{1}$$

The quantity given by (1) can be used to evaluate the percentage of error that occurs while using predicted data instead of original data. The ratio below is used for the purpose

$$R = \sqrt{\frac{\sum_{t=0}^{n}\left(y(t)-\hat{y}(t)\right)^2}{n}} \Bigg/ \sqrt{\frac{\sum_{t=0}^{n}\left(y(t)\right)^2}{n}} \quad . \tag{2}$$

Equation (2) gives the magnitude of the error that occurs when the estimated values are used in place of the original values.

4 Examples

In this section, we proved by the mean of examples that the ψ-Caputo fractional differential equation can fit the data better than the classical differential equation, provided that a suitable kernel ψ is selected.

4.1 Radioactivity experiment

Atoms present in the matter are characterized by their nuclear state. The nuclear state of an element can change due to some chemical reaction or even when it is heated to some extent. Radioactivity studies focus on nuclear state change. In general, the nuclear state change of an atom produces either electromagnetic radiation or some particles. In the process of nuclear state change, the number of atoms involve in the process may change as well. Hence one can observe for instance a fusion of two neutrons with two protons or fission of nucleus. The known forms of radiation are Alpha, Beta and Gamma. An experiment found in [22], studies the radioactivity of Silver and Aluminum ($^{110}\mathbf{A}g$, $^{108}\mathbf{A}g$ and $^{28}\mathbf{A}l$) , which represents a decay model, since particles are emitted over the time. The experimental data of $^{28}\mathbf{A}l$ is found in [6, page 6]. A scatter plot of the data shows that an exponential decay model is suitable to fit the data set. The analytic form of the model is given by

$$\frac{dn(t)}{dt} = -\lambda n(t), \qquad t \geq 0, \tag{3}$$

The classical solution to the differential equation (3) is given by

$$n(t) = n_0 \exp\left(-\frac{t}{\tau}\right) . \tag{4}$$

Where $\lambda = \frac{1}{\tau}$, t is the time elapsed, n is the number of particles yet to disintegrate at the time t. The quantity n_0 is the initial number of particles at $t = 0$, and τ is the mean life-time. The purpose of the experiment is to estimate the value of τ that would lead to a best fit of the experimental data using formula (4). Experimental data and *Matlab* optimization routine *lsqcurvefit* [23], returned $\lambda = 0.0121 \leftrightarrow \tau = 82.6446$ as the best value that would help in minimizing the error while fitting the original data using (4).

Let us now admit that the Radioactivity model defined above is ruled by the ψ-Caputo fractional differential equation. This equation is derived from the classical equation (3), with the assumption that the order of derivative is not integer. The fractional counterpart of (3) using ψ-Caputo derivative is given by

$${}_C D_{0^+}^{\alpha,\psi} n(t) = -\lambda n(t), \tag{5}$$

applying the fractional integration $I_{0^+}^{\alpha,\psi}$ to both sides of (5) and considering theorem (7.2) in [24], the exact solution of (5) can be written as

$$n(t) = n_0 E_\alpha\left(-\lambda\left(\psi(t)-\psi(0)\right)^\alpha\right), \tag{6}$$

where $E_\alpha(\cdot)$ is the Mittage-Leffler function defined by $E_\alpha(t)=\sum_{i=0}^{\infty}\frac{t^i}{\Gamma(\alpha i+1)}$, $t\in R$. $\alpha>0$ is the fractional order of derivative and λ is the inverse of the mean life-time τ.

Two parameters are to be optimized in equation (6). These are the fractional order of derivative α and the inverse of the mean life-time λ. Moreover, the kernel function is selected in a way to verify properties listed in definition 2.4. For each selected kernel function, corresponding optimal parameters are obtained. Table 1 is a summary of a sample of kernels, their corresponding optimal parameters and their root mean squared deviation.

Table 1. Radioactivity error comparison

	Classical model	**Kernel** $\psi(x)=x$	**Kernel** $\psi(x)=\sqrt{x+1}$	**Kernel** $\psi(x)=\ln(x+1)$
λ	0.0121	0.0314	0.1177	0.1046
α	1	0.8252	1.0571	1.6420
SSE	$1.17*10^5$	$2.9*10^3$	$8.87*10^3$	$2.65*10^4$
RMSD	67.2560	10.5557	18.4694	31.9074
R	18.58%	2.92%	5.10%	8.81%

The optimal parameters minimize the total error term when the original data is fitted by the fractional model defined by equation (6).

Fig. 1a, 1b and 1c below show the original experimental data, the fitted model using classical and ψ-Caputo fractional order of derivative approaches with different kernels as defined in Table 1. It is observable that in general fitting the data with the fractional approach provides less sum of squared error than the classical approach. The performance is evaluated in Table 1.

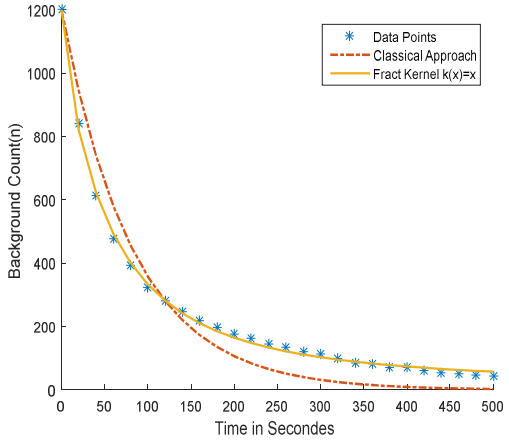

Fig. 1a

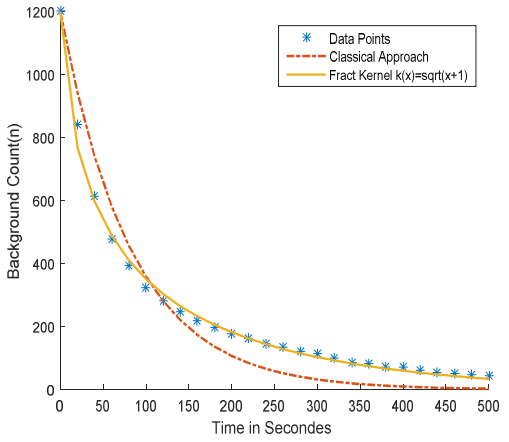

Fig. 1b

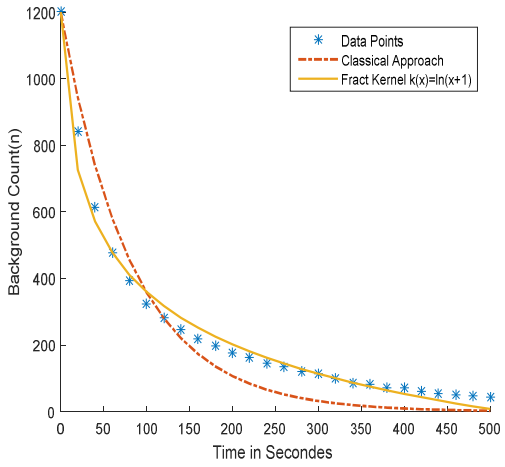

Fig. 1c

Fig. 1a-c. Radioactivity decay with different kernels

From both Table 1 and Fig. 1. It is observable that the ψ-Caputo fractional differential equation has an error rate smaller than the classical method of differential equations.

4.2 RC circuit

The RC circuit is a well known experimental setting that is used to measure the charging and discharging time of a capacitor, C. Fig. 2 shows the experimental setting of an RC circuit.

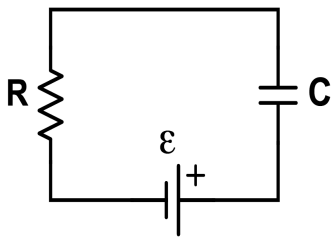

Fig. 2a. RC in charging mode

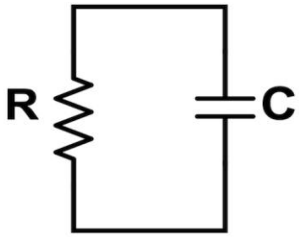

Fig. 2b. RC in discharging mode

In the charging mode, see Fig. 2a, The capacitor C, the resistor R, and the DC/voltage source ε are connected. This leads to an accumulation of electric charges in the capacitor C over the time. In the discharging mode, the potential source is removed from the circuit. Only a capacitor initially full of electric charges Q is directly connected to a resistor R. Such setting leads to a loss of electric charges over the time. An experimental data set of both charging and discharging of a RC circuit is found at [25].

Any electric component that resists to electric charges flow in an electric circuit is called resistor. Such component usually produces heat as a results of it electric resistance. That is the principle used in electric heater, iron for instance. A resistor of resistance R in a circuit obeys to the Ohm's law

$$V = I \times R \ , \tag{7}$$

where V is the voltage applied to the resistor, I is the rate at which electric charges flow through the resistor, and R is a constant called the resistance.

A capacitor is any electric circuit component with the ability to store from a circuit connected to a voltage source and to supply electric charge Q, in a circuit without voltage source. When it is attached to a resistor, the capacitor will push the electric charges through the resistor, creating electric current. Capacitors obey the equation

$$V = \frac{Q}{C}, \tag{8}$$

where Q is the charge stored in the capacitor, C is a constant called the capacitance of the capacitor, and V is the resulting voltage.

Setting the formulas (7) and (8) for V equal to each other gives

$$\frac{Q}{C} = I \times R \cdot \tag{9}$$

Moreover, the rate I at which charges flow through the resistor is the same as the rate at which charges flow out of the capacitor, so

$$-\frac{dQ}{dt} = I \cdot \tag{10}$$

A combination of (9) and (10) introduces the RC circuit model which can be written as

$$\frac{dQ}{dt} = -\frac{1}{RC}Q \cdot \tag{11}$$

It is familiar that the solution for the differential equation (11) is given by

$$Q = Q_0 e^{\frac{-1}{RC}t}, \tag{12}$$

where Q_0 is the amount held in the capacitor at $t = 0$, using the fact that $Q = CV$, (12) becomes

$$V = V_0 e^{\frac{-1}{RC}t}. \tag{13}$$

It is common in RC circuit to estimate the capacitance C. Experimental data retrieved from [25], has the following initial parameters. The resistance $R = 2200\Omega$ and the potential of the voltage source $V_0 = 4.897$. The purpose of the experiment is to estimate the value of the capacitance C found in (13). Based on experimental data and optimization routine, the value of C=0.0057 appears to be the best that would help minimizing the total error if equation (13) is used to fit the original data.

Let us assume now that the RC circuit model (11) is built based on a fractional differential equation. Such equation can be written using the ψ-Caputo derivative as

$$\begin{cases} {}_C D_{0^+}^{\alpha,\psi} Q(t) = -\frac{1}{RC} Q(t) \\ Q(0) = Q_0 \end{cases}. \tag{14}$$

Applying $I_{0^+}^{\alpha,\psi}$ to both sides of (14) and using theorem (7.2) from [24], the exact solution of (14) can be written as

$$Q(t) = Q_0 E_\alpha\left(-\frac{1}{RC}(\psi(t)-\psi(0))^\alpha\right), \tag{15}$$

using the relation $Q = CV$, equation (15) becomes,

$$V(t) = V_0 E_\alpha\left(-\frac{1}{RC}(\psi(t)-\psi(0))^\alpha\right). \tag{16}$$

Different kernel functions ψ are considered for experiment. Table 2 summarizes the experiments results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α.

Table 2. RC discharging performances

	Classical model	**Kernel** $\psi(x)=x$	**Kernel** $\psi(x)=\sqrt{x+1}$	**Kernel** $\psi(x)=\ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9871	1.1798	1.5855
SSE	0.8503	0.0098	0.7431	1.5906
RMSD	0.0922	0.0099	0.0862	0.1261
R	5.43%	0.58%	5.08%	7.43%

It is observable from Table 2 that the kernel function $\psi(x)=x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x)=\ln(x+1)$, produces an error rate slightly greater than the rate of the classical approach. The other kernel namely $\psi(x)=\sqrt{x+1}$ has an error rate which is almost equal to the 5% obtained in the classical approach.

Fig. 3 is a set of graphs displaying the behavior of the original data and the experimental results.

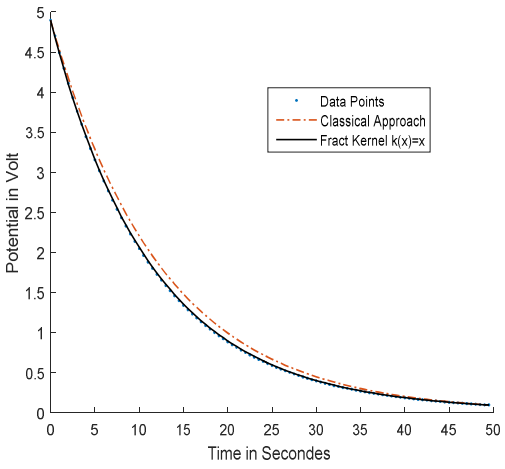

Fig. 3a

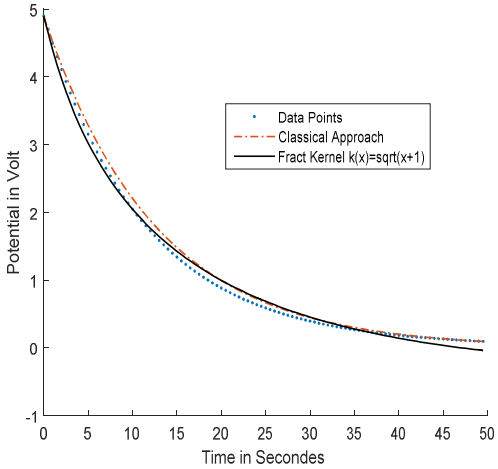

Fig. 3b

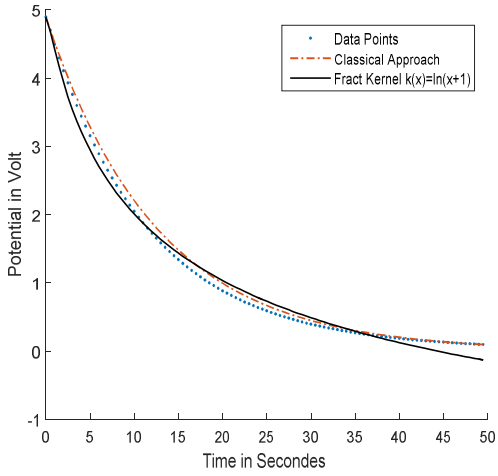

Fig. 3c

Fig. 3a-c. RC discharging with different kernels

For the charging mode the RC circuit has the following elements, a capacitance C, the resistor R and a potential source ε, that maintains the potential leading to the accumulation of electrical charges in the circuit (see Fig. 2a). The Kirchhoff's current law of the circuit is written as

$$V_0 + IR - \frac{Q}{C} = 0 \ , \tag{17}$$

Since $Q'(t) = I(t)$ the charging differential equation model is derived from (17) as

$$V_0 + \frac{dQ}{dt}R - \frac{Q}{C} = 0 \ . \tag{18}$$

The exact solution of the ordinary differential equation (18) is given by

$$Q(t) = CV_0\left(1 - e^{\frac{-1}{RC}t}\right), \tag{19}$$

which is equivalent to

$$V(t) = V_0\left(1 - e^{\frac{-1}{RC}t}\right) \ . \tag{20}$$

The use of experimental data retrieved from [25], gives a same value of C=0.0057 as obtained in the discharging process.

To solve the fractional counterpart model equation derived from (18), let us defined (18) using the ψ- Caputo derivative as

$$\begin{cases} {}_{C}D_{0^+}^{\alpha,\psi}Q(t) = \dfrac{1}{RC}Q(t) - \dfrac{V_0}{R}, \\ Q(0) = Q_0 \end{cases} \tag{21}$$

Applying $I_{0^+}^{\alpha,\psi}$ to both sides of (21) and using theorem (7.2) in [24] the exact solution of (21) can be written as

$$Q(t) = CV_0\left(1 - E_\alpha\left(\frac{-1}{RC}(\psi(t) - \psi(0))^\alpha\right)\right), \tag{22}$$

which is equivalent to

$$V(t) = V_0\left(1 - E_\alpha\left(\frac{-1}{RC}(\psi(t) - \psi(0))^\alpha\right)\right). \tag{23}$$

Different kernel functions ψ are considered for experiment. Table 3 summarizes the experimental results providing optimal values of the parameters involved alongside the error rate obtained when fitting the original data by a model. The estimated parameters in this experiment are the capacitance value C and the fractional order of derivative α.

Table 3. RC charging performances

	Classical model	**Kernel** $\psi(x) = x$	**Kernel** $\psi(x) = \sqrt{x+1}$	**Kernel** $\psi(x) = \ln(x+1)$
C	0.0057	0.0051	0.0014	0.0018
α	1	0.9962	1.1864	1.5904
SSE	1.2646	0.0127	0.7277	1.5925
RMSD	0.1125	0.0113	0.0853	0.1262
R	2.83%	0.28%	2.15%	3.18%

It is observable from Table 3 that the kernel function $\psi(x) = x$ performs better than others. Indeed it requires an optimal capacitance which is almost equal to the capacitance value of the classical approach, a fractional order of derivative close to 1, and a 0% error rate. The kernel $\psi(x) = \sqrt{x+1}$ also performs better than the classical approach. However using the kernel $\psi(x) = \ln(x+1)$, leads to a solution which is a bit weaker than the classical approach solution.

Fig. 4 is a set of graphs displaying the behavior of the original data and the experimental results of the charging process.

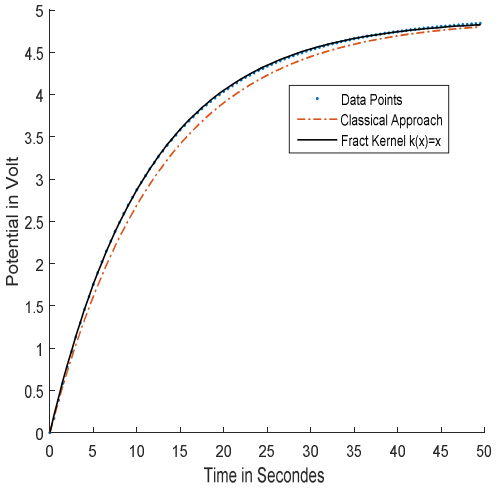

Fig. 4a

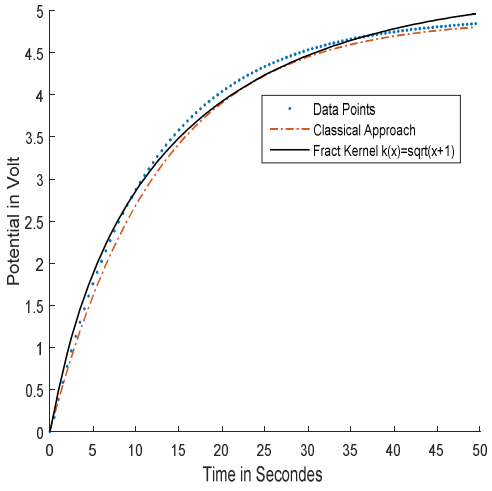

Fig. 4b

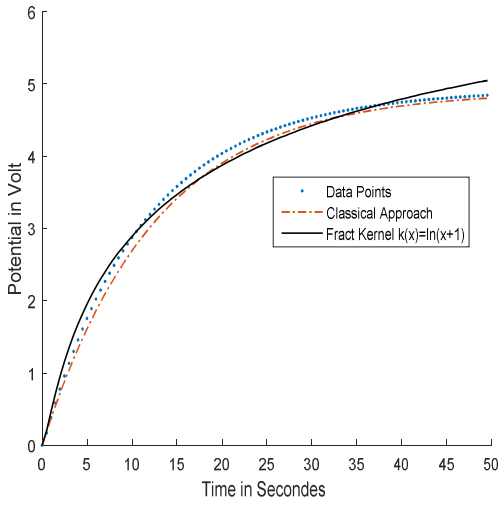

Fig. 4c

Fig. 4a-c. RC charging performances

5 Discussion

1) Interpreting the meaning of a fractional derivative is often difficult. In the physical sense for instance, the first derivative found in the RC and radioactivity differential equations represent the speed at which charges flow in the circuit and the number of radioactive element decrease respectively. It is therefore important to give a meaning to the fractional derivative order α, found in the counterpart fractional differential equation of a given problem. It is not easy to provide a meaning to α, without considering the classical model of the same problem. It is worth note the fact that the fractional order derivative of the problem studied in this paper is a value approaching one. Moreover, if $\alpha = 1$, then the Mittag Leffler function coincides with the classical exponential function. That is $E_\alpha(t) = \sum_{i=0}^{\infty} \frac{t^i}{\Gamma(\alpha i + 1)} = e^t$, $t \in R$. It follows that the solution to the classical approach coincides exactly with the solution of to the fractional approach. The statement above shows that fractional derivative is effective and meaningful.

2) In this work, we considered differential equations whose fractional model are easily obtained from their classical counterpart. However, this should not be used as a property. The Carleman Embedding technique [26], which is applicable to classical linear systems, is not always applicable to fractional system. For instance West. B [27] proposed a function to be the exact solution to the fractional logistic equation , his allegation was later proven incorrect by Area et al. [28] because the function proposed by West was derived using the Carleman Embedding technique.

6 Conclusion

The goal of this work was to prove the efficiency of the ψ-Caputo method in solving fractional differential equations upon the classical approach. This was done through experimental study. Several kernels were used. It appeared in general that the ψ-Caputo method produced a smaller magnitude error than the classical method, provided that the right kernel is selected. Moreover there exists no rule based on which the suitable kernel is selected. The selection is done based on the criteria in definition 2.3 and the variation of experimental data.

Competing Interests

Authors have declared that no competing interests exist.

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