



## Some Results for Feng-Liu type Mappings in Modular Metric Spaces

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### Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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## Abstract

In this paper we investigate some fixed point results of Feng and Liu type multi-valued mappings in modular metric spaces. We also give an example to show that these results are more general versions of Feng and Liu's theorem.

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## 1 Introduction

Fixed point theory is one of the very popular tools in various fields. It has been used to prove the existence and uniqueness of solutions for some integral and differential equations. Since Banach introduced this theory in 1922 [1], it has been extended and generalized by several authors due to its simplicity and practicability. Caristi type fixed point theorem is one of these generalizations. It is a modification of  $\varepsilon$ -variational principle of Ekeland [2]. It is crucial in nonlinear analysis, in particular, optimization, variational inequalities, differential equations, and control theory. Other than that, this theorem has been extended to multi-valued mappings by Nadler [3]. Since it is very important in several fields of mathematics. Feng and Liu gave a generalization of multivalued contractive mappings and multivalued Caristi type mappings [4]. Also, Khojasteh, Karapinar, and Khandani investigated Caristi type fixed point theorems for multivalued mappings in metric spaces and they gave an application in functional equations [5].

The notion of modular space was introduced by Nakano [6] and was intensively developed by Koshi, Shimogaki, Yamamuro (see [7, 8]) and others. A lot of mathematicians are interested in a fixed point of modular space. In 1990 Khamsi, Kozlavski and Reich introduced modular function spaces. In 2008, Chistyakov introduced the notion of modular metric space generated by F-modular and developed the theory of this space [9], on the same idea he defined the notion of a modular on an arbitrary set and developed the theory of metric space generated by modular such called as modular metric spaces in 2010 [10]. He also investigated topology and convergence of the modular metric spaces. Kılınç and Alaca [11] defined  $(\varepsilon, k)$ -uniformly locally contractive mappings and  $\eta$ -chainable concept and proved a fixed point theorem for these concepts in complete modular metric spaces. Kılınç and Alaca [12] proved that two main fixed point theorems for commuting mappings in modular metric spaces. Recently, many authors [13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26], studied on different fixed point results for modular metric spaces. In 2014 Khamsi and Abdou investigated Hausdorff metric modular in modular metric spaces [27], and proved fixed point theorem for multi-valued mappings. Chaipunya et. al. gave some fixed point results of multivalued mappings in modular metric spaces [28]. Turkoglu and Kılınç gave modular metric versions of Khojasteh, Karapinar, and Khandani's results [29].

In this paper we try to give modular metric versions of Feng and Liu's results which are more general than their metric counterparts. Readers must notice that modular convergence is weaker than general metric convergence and modular topology is coarser than metric topology on  $X_w$ .

## 2 Preliminaries

### 2.1 Metric Modulars

In this section, we will give some basic concepts and definitions about modular metric spaces which are useful to show the main results.

**Definition 2.1** [[10], Definition 2.1] Let  $X$  be a nonempty set, a function  $w : (0, \infty) \times X \times X \rightarrow [0, \infty]$  is said to be a metric modular on  $X$  if satisfying, for all  $x, y, z \in X$  the following condition holds:

- (i)  $w_\lambda(x, y) = 0$  for all  $\lambda > 0 \Leftrightarrow x = y$ ;
- (ii)  $w_\lambda(x, y) = w_\lambda(y, x)$  for all  $\lambda > 0$ ;
- (iii)  $w_{\lambda+\mu}(x, y) \leq w_\lambda(x, z) + w_\mu(z, y)$  for all  $\lambda, \mu > 0$ .

If instead of (i), we have only the condition

- ( $\hat{i}$ )  $w_\lambda(x, x) = 0$  for all  $\lambda > 0$ , then  $w$  is said to be a (metric) pseudomodular on  $X$ .

**The main property of a metric modular** [[10]]  $w$  on a set  $X$  is the following: given  $x, y \in X$ , the function  $0 < \lambda \mapsto w_\lambda(x, y) \in [0, \infty]$  is nonincreasing on  $(0, \infty)$ . In fact, if  $0 < \mu < \lambda$ , then (iii), (i) and (ii) imply

$$w_\lambda(x, y) \leq w_{\lambda-\mu}(x, x) + w_\mu(x, y) = w_\mu(x, y).$$

It follows that at each point  $\lambda > 0$  the right limit  $w_{\lambda+0}(x, y) = \lim_{\mu \rightarrow \lambda+0} w_\mu(x, y)$  and the left limit  $w_{\lambda-0}(x, y) = \lim_{\varepsilon \rightarrow +0} w_{\lambda-\varepsilon}(x, y)$  exist in  $[0, \infty]$  and the following two inequalities hold:

$$w_{\lambda+0}(x, y) \leq w_\lambda(x, y) \leq w_{\lambda-0}(x, y).$$

## 2.2 Modular convergence and modular topology

In this section we give modular topology and modular convergence in the sense of Chistyakov. Readers can also see [30] for further informations. But first of all we start with metrizable of modular spaces.

**Definition 2.2** [30] Let  $w$  be a pseudomodular on a set  $X$ . Then  $w$  induces an equivalence relation  $\overset{w}{\sim}$  on  $X$  as follows:

$$\text{let } x, y \in X, x \overset{w}{\sim} y \text{ iff } w_\lambda(x, y) < \infty \text{ for some } \lambda > 0$$

Then for a fixed  $x_0 \in X$ , we can define an equivalence class as follows:

$$[x_0] = \{x \in X : \exists \lambda > 0 \text{ such that } w_\lambda(x, x_0) < \infty\}$$

This class is called a modular space (around  $x_0$ ) and shown as  $X_w(x_0)$ .

Now we can define metric on  $X_w(x_0)$  as follows:

**Definition 2.3** [30] Let  $w$  be a pseudomodular on  $X$ . Set

$$d_w^0(x, y) = \inf \{\lambda > 0 : w_\lambda(x, y) \leq \lambda\}$$

where  $x, y \in X$ , then  $d_w^0$  is an extended pseudometric on  $X$  and if  $d_w^0(x, y) < \infty$  is equivalent to  $x \overset{w}{\sim} y$ ,  $d_w^0$  is a pseudometric on  $X_w(x_0)$ .

Kumam and Chaipunya [31] showed that if  $X_w(x_0) = X$  for all  $x_0 \in X$ , then we write  $X_w$  in place of  $X_w(x_0)$ .

**Theorem 2.1.** [2.1] [30] Given a sequence  $(x_n) \subset X$  and  $x \in X$ , we have:

$$x_n \rightarrow x \text{ if and only if } w_\lambda(x_n, x) \rightarrow 0 \text{ for all } \lambda > 0 \tag{2.1}$$

Moreover, if  $w$  is convex, then (2.1) is also equivalent to  $d_w(x_n, x) \rightarrow 0$ , and if  $w$  is a modular on  $X$ , then the limit is unique.

**Definition 2.4** [30] Given  $x \in X$  and  $r > 0$ , the open ball of radius  $r$  centered at  $x$  is the set

$$B(x, r) \equiv B_w^{d_w^0}(x, r) = \{y \in X : d_w^0(x, y) < r\}$$

and a nonempty set  $U \subset X$  is open if for every  $x \in U$ , there exists  $r > 0$  such that  $B(x, r) \subset U$ .

Now we can give metric topology  $\tau(d_w^0)$  on  $X_w$ :  $U \in \tau(d_w^0)$  iff  $U^c$  is closed with respect to the metric convergence. Chistyakov gave modular open sets as follows which he called  $w$ -entourages (or modular entourages):

**Definition 2.5** [30] Given  $\lambda, \mu > 0$  and  $x \in X_w$ , the  $w$ -entourage about  $x$  relative to  $\lambda$  and  $\mu$  is the set

$$B_{\lambda, \mu}(x) \equiv B_{\lambda, \mu}^w(x) = \{y \in X_w : w_\lambda(x, y) < \mu\}$$

Thus we can give the caractarizationof open sets in terms of w:

$$U \in \tau(d_w^0) \Leftrightarrow \forall x \in X_w \exists \mu > 0 \text{ such that } B_{\mu,\mu}(x) \subset U$$

**Definition 2.6 [30]** A nonempty set  $U \subset X$  said to be  $w$ -open if for every  $x \in U$  and  $\lambda > 0$  there is  $\mu > 0$  such that  $B_{\lambda,\mu}(x) \subset U$ . If  $\tau(w)$  is a family of all  $w$ -open sets on  $X_w$ , then  $\tau(w)$  is called  $w$ -topology (or modular topology) on  $X_w$ .

**Definition 2.7 [[18]]** Let  $X_w$  be a modular metric space. Then following definitions exist:

- (1) The sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X_w$  is said to be convergent to  $x \in X_w$  if  $w_\lambda(x_n, x) \rightarrow 0$ , as  $n \rightarrow \infty$  for some  $\lambda > 0$
- (2) The sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X_w$  is said to be Cauchy if  $w_\lambda(x_m, x_n) \rightarrow 0$ , as  $m, n \rightarrow \infty$  for some  $\lambda > 0$
- (3) A subset  $C$  of  $X_w$  is said to be closed if the limit of a convergent sequence of  $C$  always belong to  $C$ .
- (4) A subset  $C$  of  $X_w$  is said to be complete if any Cauchy sequence in  $C$  is a convergent sequence and its limit is in  $C$ .
- (5) A subset  $C$  of  $X_w$  is said to be  $w$ -bounded if

$$\delta_w(C) = \sup \{w_\lambda(x, y); x, y \in C\} < \infty.$$

- (6) A subset  $C$  of  $X_w$  is said to be  $w$ -compact if for any  $(x_n)$  in  $C$  there exists a subset sequence  $(x_{n_k})$  and  $x \in C$  such that  $w_\lambda(x_{n_k}, x) \rightarrow 0$
- (7)  $w$  is said to satisfy the Fatou property if and only if for any sequence  $(x_n)_{n \in \mathbb{N}}$  in  $X_w$   $w$ -convergent to  $x$ , we have

$$w_\lambda(x, y) \leq \liminf_{n \rightarrow \infty} w_\lambda(x_n, y),$$

- (8)  $w$  is said to satisfy  $\Delta_2$ -condition iff  $\lim_{n \rightarrow \infty} w_\lambda(x_n, x) = 0$  for some  $\lambda > 0$  implies  $\lim_{n \rightarrow \infty} w_\lambda(x_n, x) = 0$  for all  $\lambda > 0$ .

### 3 Main Results

Now we we will give our main results. Before we start let us define a set  $I_b^x$  as follows.

Let  $T : X_w \rightarrow N(X_w)$  be a multi-valued mapping. Define the function  $f_\lambda : X_w \rightarrow \mathbb{R}_+$  as  $f(x) = w_\lambda(x, T(x))$ .

For a constant  $b$  ( $b \in (0, 1)$ ), define the set  $I_b^x \subset X_w$  as

$$I_b^x = \{y \in T(x) \mid bw_\lambda(x, y) \leq w_\lambda(x, T(x))\}$$

Let us consider the set  $I_b^x$ : 1 - a) Let  $x \in T(x)$  and  $y = x$ , then

$$\begin{aligned} bw_\lambda(y, x) &= bw_\lambda(x, x) \leq w_\lambda(x, T(x))0 \\ w_\lambda(x, T(x)) &= 0 \end{aligned} \leq$$

1 - b) Let  $x, y \in T(x)$  and  $y \neq x$ , then  $w_\lambda(x, T(x)) = 0$  but  $w_\lambda(y, x) \neq 0$  which means the inequality  $bw_\lambda(x, y) \leq w_\lambda(x, T(x))$  couldn't be satisfied for any  $b \in (0, 1)$ , that is the set  $I_b^x = \{x\}$  if  $x \in T(x)$

2) Let  $x \notin Tx$ , then  $y \in T(x)$  for some  $y \neq x$ . When we use the definition of modular distance we get

$$w_\lambda(x, y) \leq w_\lambda(x, T(x)) + \varepsilon$$

for  $\varepsilon > 0$ . If we choose  $\varepsilon = (1 - b) w_\lambda(x, y)$ , we get

$$\begin{aligned} w_\lambda(x, y) &\leq w_\lambda(x, T(x)) + (1 - b) w_\lambda(x, y) \\ bw_\lambda(x, y) &\leq w_\lambda(x, T(x)) \end{aligned}$$

which means  $y \in I_b^x$ . Thus  $I_b^x \neq \emptyset$

Now let us give our main theorem of this section.

**Theorem 3.1** *Let  $X_w$  be a complete modular metric space induced by  $w$ ,  $T : X_w \rightarrow C(X_w)$  be a multi-valued mapping,  $w_\lambda(x, Tx) < \infty$  and  $w$  satisfies  $\Delta_2$ -condition. If there exists a constant  $c \in (0, 1)$  such that for any  $x \in X_w$  there is  $y \in I_b^x$  satisfying*

$$w_\lambda(y, T(y)) \leq cw_\lambda(x, y)$$

and then  $T$  has a fixed point in  $X_w$  provided  $c < b$  and  $f_\lambda$  is lower semi-continuous for some  $\lambda > 0$ .

*Proof.* Since  $T(x) \in C(X_w)$  for any  $x \in X_w$ ,  $I_b^x$  is nonempty for any constant  $b \in (0, 1)$ . For any initial point  $x_0 \in X_w$ , there exists  $x_1 \in I_b^{x_0}$  such that

$$w_\lambda(x_1, T(x_1)) \leq cw_\lambda(x_0, x_1)$$

for some  $\lambda > 0$  and for  $x_1 \in X_w$ , there exists  $x_2 \in I_b^{x_1}$  satisfying

$$w_\lambda(x_2, T(x_2)) \leq cw_\lambda(x_1, x_2)$$

for some  $\lambda > 0$ . Continuing this iteration we can get a sequence  $(x_n)$ , where  $x_{n+1} \in I_b^{x_n}$

$$w_\lambda(x_{n+1}, T(x_{n+1})) \leq cw_\lambda(x_n, x_{n+1}), \quad n = 0, 1, 2, \dots$$

for some  $\lambda > 0$ . On the other hand  $x_{n+1} \in I_b^{x_n}$  implies that

$$bw_\lambda(x_n, x_{n+1}) \leq w_\lambda(x_n, T(x_n)), \quad n = 0, 1, 2, \dots$$

for some  $\lambda > 0$ . From These two inequalities above we can write

$$\begin{aligned} w_\lambda(x_{n+1}, T(x_{n+1})) &\leq cw_\lambda(x_n, x_{n+1}) \leq \frac{c}{b} w_\lambda(x_n, T(x_n)) \\ w_\lambda(x_{n+1}, T(x_{n+1})) &\leq \frac{c}{b} w_\lambda(x_n, T(x_n)) \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} w_\lambda(x_n, x_{n+1}) &\leq \frac{1}{b} w_\lambda(x_n, T(x_n)) \leq \frac{c}{b} w_\lambda(x_n, x_{n-1}) \\ w_\lambda(x_n, x_{n+1}) &\leq \frac{c}{b} w_\lambda(x_n, x_{n-1}) \end{aligned} \tag{3.2}$$

for some  $\lambda > 0$ . Using the inequality (3.2) successively we get

$$w_\lambda(x_n, x_{n+1}) \leq \frac{c}{b} w_\lambda(x_n, x_{n-1}) \leq \dots \leq \left(\frac{c}{b}\right)^n w_\lambda(x_1, x_0), \quad n = 0, 1, 2, \dots$$

and using the inequality (3.1) successively, we get

$$w_\lambda(x_n, T(x_n)) \leq \frac{c}{b} w_\lambda(x_{n-1}, T(x_{n-1})) \leq \dots \leq \left(\frac{c}{b}\right)^n w_\lambda(T(x_0), x_0), \quad n = 0, 1, 2, \dots$$

for all  $\lambda > 0$ . Then, for  $m, n \in \mathbb{N}$ ,  $m > n$

$$\begin{aligned} w_\lambda(x_n, x_m) &\leq w_{\frac{\lambda}{m-n}}(x_m, x_{m-1}) + \dots + w_{\frac{\lambda}{m-n}}(x_{n+1}, x_n) \\ &\leq \left(\frac{c}{b}\right)^{m-1} w_{\frac{\lambda}{m-n}}(x_0, x_1) + \dots + \left(\frac{c}{b}\right)^n w_{\frac{\lambda}{m-n}}(x_0, x_1) \end{aligned}$$

Since  $c < b$ ,  $\lim_{n \rightarrow \infty} (\frac{c}{b})^n = 0$  and since  $0 < w_\lambda(x_n, x_{n+1}) < \infty$  we get  $\lim_{n \rightarrow \infty} w_\lambda(x_n, x_{n+1}) = 0$  Since  $w$  satisfies  $\Delta_2$ -condition, if  $\lim_{n \rightarrow \infty} w_\lambda(x_n, x_{n+1}) = 0$ , then  $\lim_{n \rightarrow \infty} w_{\frac{\lambda}{2}}(x_n, x_{n+1}) = 0$ . If we continue this iteration we get  $\lim_{n \rightarrow \infty} w_{\frac{\lambda}{2^k}}(x_n, x_{n+1}) = 0$  for some  $k \in \mathbb{N}$ . If we choose  $k \in \mathbb{N}$  such that  $2^k < m - n$ , then from the main property of modular metric we get  $\lim_{n \rightarrow \infty} w_{\frac{\lambda}{m-n}}(x_n, x_{n+1}) = 0$  for some  $\lambda > 0$ .

If we take the limit as  $n \rightarrow \infty$ , we get

$$\begin{aligned} \lim_{n \rightarrow \infty} w_\lambda(x_n, x_m) &\leq \lim_{n \rightarrow \infty} \frac{(\frac{c}{b})^n}{1 - (\frac{c}{b})} w_{\frac{\lambda}{m-n}}(x_0, x_1) \\ \lim_{n \rightarrow \infty} w_\lambda(x_n, x_m) &= 0 \end{aligned}$$

which means that  $(x_n)$  is a Cauchy sequence. Since  $w$  satisfies  $\Delta_2$ -condition  $X_w$  is complete, and there exists  $x \in X_w$  such that  $(x_n)$  converges to  $x$ . On the other hand  $\{f_\lambda(x_n)\}_{n=0}^\infty = \{w_\lambda(x_n, T(x_n))\}_{n=0}^\infty$  is decreasing. In fact

$$f_\lambda(x_n) = w_\lambda(x_n, T(x_n)) \leq \frac{c}{b} w_\lambda(x_{n-1}, T(x_{n-1})) = \frac{c}{b} f_\lambda(x_{n-1})$$

and since  $\frac{c}{b} < 1$ , then  $\{f_\lambda(x_n)\}_{n=0}^\infty$  is decreasing for some  $\lambda > 0$ .  $f_\lambda(x) = w_\lambda(x, T(x))$  is positive so  $\{f_\lambda(x_n)\}_{n=0}^\infty$  converges to 0. Since  $f$  is lower semi-continuous, we have

$$0 \leq f_\lambda(x) \leq \liminf_{n \rightarrow \infty} f_\lambda(x_n) = 0.$$

Hence  $f_\lambda(x) = 0$ . And from the closeness of  $T(x)$  we can say that  $x \in T(x)$ . And this completes the proof.  $\square$

Now let us give some remarks of this theorem.

**Remark 3.1** Theorem 1 is a generalization of Chaipunya et al.'s theorem [28].

(1) In fact, if  $T$  satisfies the conditions of Chaipunya et al.'s theorem, then;  $f$  is lower semi-continuous, which follows from the fact that  $T$ , being a multi-valued contraction, is upper semi-continuous. And for any  $x \in X_w$ ,  $y \in T(x)$ ,  $\lambda > 0$

$$w_\lambda(y, T(y)) \leq H_w(T(x), T(y)) \leq cw_\lambda(x, y).$$

Hence  $T$  satisfies conditions of theorem 1, the existence of a fixed point has been proved.

(2) The following example shows that Theorem 1 is an extension of Chaipunya et al.'s theorem [28].

Let  $X_w = \{\frac{1}{2}, \frac{1}{4}, \dots, \frac{1}{2^n}, \dots\} \cup \{0, 1\}$ ,  $w_\lambda(x, y) = \frac{1}{\lambda} |x - y|$  for  $x, y \in X_w$  and for all  $\lambda > 0$ . Specifically we choose  $\lambda = 1$ , then  $w_1(x, y) = |x - y|$  and  $X_w$  is a complete modular metric space. Define mapping  $T : X_w \rightarrow C(X_w)$  as

$$T(x) = \begin{cases} \{\frac{1}{2^{n+1}}, 1\}, & x = \frac{1}{2^n}, n = 0, 1, \dots \\ \{0, 1\}, & x = 0. \end{cases}$$

One can see that  $T$  is not a contractive mapping in Chaipunya et al.'s meaning, in fact for

$$H_w(T(\frac{1}{2^n}), T(0)) = \frac{1}{2} \geq \frac{1}{2^n} = w_1(\frac{1}{2^n}, 0), n = 1, 2, \dots$$

But one can show that

$$f(x) = w_1(x, T(x)) = \begin{cases} \frac{1}{2^{n+1}}, & x = \frac{1}{2^n}, n = 1, 2, \dots \\ 0, & x = 0, 1 \end{cases}$$

and  $f$  is continuous. Moreover there exists  $y \in I_{0,5}^x$  for any  $x \in X_w$  and

$$w_1(y, T(y)) = \frac{1}{2}w_1(x, y)$$

**Remark 3.2** Theorem 3.1 doesn't guarantee the uniqueness of a fixed point.

Now we can give a proposition after this theorem.

**Proposition 3.1** Let  $X_w$  be a complete modular metric space,  $T : X_w \rightarrow C(X_w)$  be a multi-valued mapping. If there exists a constant  $c \in (0, 1)$  such that for any  $x \in X_w, y \in T(x)$  and for all  $\lambda > 0$

$$w_\lambda(y, T(y)) \leq cw_\lambda(x, y),$$

then  $T$  has a fixed point in  $X_w$  provided  $f = w_\lambda(x, T(x))$  is lower semi-continuous.

**Remark 3.3**  $f(x)$  being lower semi-continuous play crucial role proving these theorems. Without this assumption we couldn't show that  $T$  has a fixed point even if the other assumptions hold for all  $\lambda > 0$ . Define mapping  $T : X_w \rightarrow C(X_w)$  as

$$T(x) = \begin{cases} \left\{ \frac{1}{2^{n+1}}, \frac{1}{2^{n+2}} \right\}, & x = \frac{1}{2^n}, n = 0, 1, 2, \dots \\ \{1\}, & x = 0 \end{cases}$$

and define metric modular as

$$w_\lambda(x, y) = |x - y|$$

for  $x, y \in X_w$  and  $\lambda = 1$ . It is easy to show that

$$f(x) = \begin{cases} \frac{1}{2^{n+1}}, & x = \frac{1}{2^n}, n = 0, 1, 2, \dots \\ \{1\}, & x = 0 \end{cases}$$

Hence the function  $f$  is not lower semi-continuous and although other conditions hold  $T$  has not a fixed point in  $X_w$ .

## 4 An Application to Non-homogeneous Linear Parabolic Partial Differential Equations

Poom et. al. investigated applications to differential equations for fixed point theorems in [32]. Now we try to show that these results are naturally satisfied for Feng and Liu type mappings in modular metric spaces. In [32] they gave the initial value problem for a non-homogenous linear parabolic partial differential equation as

$$f_t(x, t) = f_{xx}(x, t) + S(x, t, f(x, t), f_x(x, t)), \quad -\infty < x < \infty, \quad 0 < t \leq T \quad f(x, 0) = \phi(x) \geq 0 \quad (4.1)$$

for  $x \in X_w$ , where  $S$  is continuous,  $\phi$  continuously differentiable such that  $\phi$  and  $\phi'$  are bounded. By a solution of this problem, a function  $f = f(x, t)$  is defined on  $\mathbb{R} \times I$  where  $I = [0, T]$  satisfies following conditions:

- i)  $f, f_t, f_x, f_{xx} \in C(C(\mathbb{R} \times I))$  where  $C(\mathbb{R} \times I)$  denotes all continuous real-valued functions,
- ii)  $f, f_x \in \mathbb{R} \times I$  are bounded,
- iii)  $f_t(x, t) = f_{xx}(x, t) + S(x, t, f(x, t), f_x(x, t)), (x, t) \in \mathbb{R} \times I,$
- iv)  $f(x, 0) = \phi(x) \geq 0$  for all  $x \in \mathbb{R}$ .

The differential equation problem above, is equivalent to the following integral equation problem:

$$f(x, t) = \int_{-\infty}^{\infty} K(x - \delta, t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(x - \delta, t - u)S(\delta, u, f(\delta, u), f_x(\delta, u))d\delta du \quad (4.2)$$

for all  $x \in \mathbb{R}$  and  $0 < t \leq T$  where;

$$K(x, t) = \frac{1}{\sqrt{4\pi t}}e^{-\frac{x^2}{4t}}.$$

The problem in (4.1) has a solution iff the corresponding problem (4.2) has a solution. Let

$$B = \{f(x, t) : f, f_x \in C(R \times I), \|f\| < \infty\}$$

where

$$\|f\| = \sup_{x \in \mathbb{R}, t \in I} |f(x, t)| + \sup_{x \in \mathbb{R}, t \in I} |f_x(x, t)|.$$

Now let us define the metric modular on  $B$  as follows:

$$w_\lambda(f, g) = \frac{1}{\lambda} |f - g|$$

Clearly  $B_w$  is a  $w$ -complete modular metric spaces and is independent from its generators. Now we can give the following theorem:

**Theorem 4.1** *Let the problem*

$$\begin{aligned} f_t(x, t) &= f_{xx}(x, t) + S(x, t, f(x, t), f_x(x, t)), \quad -\infty < x < \infty, \quad 0 < t \leq T \\ f(x, 0) &= \phi(x) \geq 0 \end{aligned}$$

and assume following:

- i) For  $c > 0$  with  $|s| < c$  and  $|p| < c$  the function  $S(x, t, s, p)$  is uniformly Hölder continuous in  $x$  and  $t$  for each compact subset of  $\mathbb{R} \times I$
- ii) There exists a constant  $c_S \leq \left(T + 2\pi^{-\frac{1}{2}}T^{\frac{1}{2}}\right)^{-1} \leq q$  where  $q \in (0, 1)$  such that

$$\begin{aligned} 0 &\leq \frac{1}{\lambda} [S(x, t, s_2, p_2) - S(x, t, s_1, p_1)] \\ &\leq c_s \left[ \frac{s_2 - s_1 + p_2 - p_1}{\lambda} \right] \end{aligned}$$

for all  $(s_1, p_1), (s_2, p_2) \in \mathbb{R} \times \mathbb{R}$  with  $s_1 \leq s_2$  and  $p_1 \leq p_2$ ,

- iii)  $S$  is bounded for bounded  $s$  and  $p$ ;

Then the problem has a solution.

*Proof.* Let us choose  $x \in B_w$  is a solution of the problem above, if and only if  $x \in B_w$  is a solution integral equivalent. When we take the graph  $G$  with  $V(G) = B_w$  and  $E(G) = \{(z, v) \in B_w \times B_w : z(x, t) \leq v(x, t) \text{ and } z_x(x, t) \leq v_x(x, t) \text{ for each } (x, t) \in R \times I\}$ ,  $E(G)$  is partially ordered and  $(B_w, E(G))$  satisfy property (A). The mapping  $F : B_w \rightarrow B_w$  defined as

$$F(f)(x, t) = \int_{-\infty}^{\infty} K(x - \delta, t)\phi(\delta)d\delta + \int_0^t \int_{-\infty}^{\infty} K(x - \delta, t - u)S(\delta, u, f(\delta, u), f_x(\delta, u))d\delta du$$

for all  $x \in \mathbb{R}$  and when we solve the problem, solution give us the existence of fixed point of  $F$ . Since  $(z, v), (z_x, v_x), (F(z), F(v)), (F(z_x), F(v_x)) \in E(G)$  and from the definition of  $F$  and (ii)

$$\frac{1}{\lambda} |F(v(x, t)) - F(z(x, t))| \leq c_S w_\lambda(z, v)$$



Therefore we have

$$\begin{aligned} \frac{1}{\lambda} |(Fv)_x(x, t) - (Fz)_x(x, t)| &\leq c_S w_\lambda(z, v) \int_{-\infty}^{\infty} K(x - \delta, t) \phi(\delta) d\delta \\ &\leq 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}} c_S w_\lambda(z, v) \end{aligned}$$

Then we have

$$\begin{aligned} w_\lambda(Fz, Fv) &\leq \left(T + 2\pi^{\frac{-1}{2}} T^{\frac{1}{2}}\right) c_S w_\lambda(z, v) \\ w_\lambda(Fz, Fv) &\leq c w_\lambda(z, v), c \in (0, 1) \\ \lambda |Fz - Fv| &\leq \lambda |z - v| \end{aligned}$$

If we take these inequalities in the sense of Feng and Liu we can get

$$\begin{aligned} \lambda |v - Fv| &\leq \lambda H_d(Fz, Fv) \leq \lambda |z - v| \\ d(v, Fv) &\leq H_d(Fz, Fv) \leq d(z, v) \\ w_\lambda(v, Fv) &\leq w_\lambda(z, v) \end{aligned}$$

since we have  $bw_\lambda(z, v) \leq w_\lambda(z, F(z))$  while  $b \in (0, 1)$ . Then there exists a  $z^* \in B_w$  such that  $F(z^*) = z^*$ . Hence this gives our problem a solution.  $\square$

## 5 Conclusion

In this paper, authors aimed to find Feng -Liu type fixed point theorems in modular metric spaces. Our results were explained in four sections. In section 1, authors tried to explain the progress of fixed point theory and modular metric spaces. Also some basic studies were given for readers. Preliminaries section parts two subsections. In subsection 2.1. authors gave some basic concepts of metric modulars and modular metric spaces. In subsection 2.2. modular convergence and modular topology were investigated. In section 3, a fixed point theorem for Feng -Liu type mappings in modular metric spaces was given. Also some examples were showed to explain the results are more general than Chaipunya's results which is given in [cite]chip:multi;cite;. Some remarks were also introduced. In section 4, an application of our results to non-homogeneous linear parabolic partial differential equations were given.

## Competing Interests

Authors have declared that no competing interests exist.

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