Article

Repeated sums and binomial coefficients

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Abstract: Binomial coefficients have been used for centuries in a variety of fields and have accumulated numerous definitions. In this paper, we introduce a new way of defining binomial coefficients as repeated sums of ones. A multitude of binomial coefficient identities will be shown in order to prove this definition. Using this new definition, we simplify some particular sums such as the repeated Harmonic sum and the repeated Binomial-Harmonic sum. We derive formulae for simplifying general *repeated sums* as well as a variant containing binomial coefficients. Additionally, we study the *m*-th difference of a sequence and show how sequences whose *m*-th difference is constant can be related to binomial coefficients.

1. Introduction

The author is interested in the study of the various types of repetitive sums, that is, developing formulae and identities to better understand these types of sums as well as to simplify the way we work with them. Previously, the author has presented a set of useful formulae for two types of repetitive sums; recurrent sums [1] and multiple sums [2]. In this article, we aim to do the same for a third type of repetitive sums, the repeated sums. For the first two types, partitions were key for developing a reduction theorem to simplify these sums. For this type of sums, binomial coefficients are key. Hence, we first need to develop a set of binomial coefficient identities as well as we need to introduce a new definition for binomial coefficients.

Binomial coefficients appear in many distinct fields of mathematics (including probability, combinatorics, analysis, etc.) and, therefore, they can be defined in many different ways. In 1654, in his article "Traité du triangle arithmétique" [3,4], Pascal presented what is now known as Pascal's triangle and which constituted the first definition of binomial coefficients.

The binomial coefficient $\binom{n}{k}$ represents the element in the *k*-th column and *n*-th row of Pascalás triangle. They can also be defined by the recurrent relation used to construct Pascal's triangle which we will call the Pascal's triangle identity and which states that $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. The notation $\binom{n}{k}$ was introduced by Euler in the 18th century. Alternatively, binomial coefficients can be denoted as C_n^k. This second notation, introduced in the 19th century, is mostly used in combinatorics where binomial coefficients are defined as the number of ways of choosing an unordered subset of *k* elements from a set of *n* elements.

Binomial coefficients are also intensively used in probability calculation for binomial distributions. Let *X* be a random variable, let *n* be the number of experiments, let *k* be an integer, and let *p* represent the probability of success in a single experiment, the binomial distribution formula is given by $P(X = k) = C_n^k p^k (1 - p)^{n-k}$.

Additional definitions of these coefficients appear in analysis and calculus. First, they appear in the binomial theorem as well as in the generalized binomial theorem presented by Newton [5,6],

$$(u + v)^n = \sum_{k=0}^{n} \binom{n}{k} u^k v^{n-k}$$

and

$$(u_1 + \cdots + u_m)^n = \sum_{k_1 + \cdots + k_m = n} \binom{n}{k_m}\binom{n - k_m}{k_{m-1}}\cdots\binom{n - k_m - \cdots - k_1}{k_2} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}.$$

From the simple case of the theorem, the binomial coefficient $\binom{n}{k}$ can be defined as the coefficient of $u^k v^{n-k}$ in the expansion of $(u+v)^n$. Likewise, $\binom{n}{k}$ appears in Leibniz's formula for the n-th derivative of a product [7–9],

$$(uv)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} u^{(k)} v^{(n-k)}.$$

Hence, Leibniz's rule defines the binomial coefficient $\binom{n}{k}$ as the coefficient of $u^{(k)} v^{(n-k)}$ in the expansion of $(uv)^{(n)}$. These coefficients even appear in number theory: In [1,2], binomial coefficients are defined as a sum over partitions of an integer m in several ways. And, finally, binomial coefficients can be defined in terms of factorials: $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Remark 1. Tables of binomial coefficients can be found in [10–12].

In this study, we propose a new definition: we define binomial coefficients as repeated sums (Section 3). This definition is key for the determination of a reduction formula for repeated sums which allows the reduction of such sums into simple non-repetitive sums. In Section 3, we also present formulas for the m-th difference of a sequence as well as we show the link between sequences with a constant m-th difference and binomial coefficients. In Section 2, we present some formulae for binomial sums and repeated binomial sums which are needed to prove the definition. In Section 4, this new definition is used to derive results related to the harmonic sum such as an expression for the repeated harmonic sum in terms of the simple harmonic sum. We also derive expressions for a modified version of the repeated harmonic sum which we will refer to as the repeated binomial-harmonic sum. Then, in Section 5, this definition as well as the results derived will be utilized to develop a reduction formula for the repeated sum of any sequence a_N. A similar expression is also derived for a modified version of repeated sums which we will refer to as the repeated "binomial–sequence" sum.

2. Sums of binomial coefficients

In this section, we prove some formulas related to the sums and repeated sums of binomial coefficients. These formulas are needed in order to produce the proposed definition for binomial coefficients. They are also essential for developing the harmonic sum identities later on. Furthermore, they are crucial for developing an expression for repeated sums in terms of single sums.

2.1. Binomial sum

We begin by determining a formula for the sum of binomial coefficients.

Theorem 1. *For any $k, q, n \in \mathbb{N}$ such that $n \geq q$, we have*

$$\sum_{i=q}^{n} \binom{i}{k} = \binom{n+1}{k+1} - \binom{q}{k+1}.$$

Remark 2. The coefficients of the form $\binom{n}{k}$ where $n < k$, being undefined according to the factorial definition, are supposed zero. Hence, if $q < k$, we can start the sum at k.

Proof. Base case: Verify the statement is true for $n = q$.
Using Pascal's Triangle identity,

$$\sum_{i=q}^{q} \binom{i}{k} = \binom{q}{k} = \binom{q+1}{k+1} - \binom{q}{k+1}.$$

Induction hypothesis: Assume the statement is true until n, i.e.,

$$\sum_{i=q}^{n} \binom{i}{k} = \binom{n+1}{k+1} - \binom{q}{k+1}.$$

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Induction step: We will show that this statement is true for $(n+1)$.

We have to show the following statement to be true:

$$\sum_{i=q}^{n+1}\binom{i}{k} = \binom{n+2}{k+1} - \binom{q}{k+1}.$$

$$\sum_{i=q}^{n+1}\binom{i}{k} = \binom{n+1}{k} + \sum_{i=q}^{n}\binom{i}{k} = \binom{n+1}{k} + \binom{n+1}{k+1} - \binom{q}{k+1}.$$

From Pascal's Triangle identity, we get the case for $(n+1)$. $\square$

Corollary 1. *For q = 0 or q = k, Theorem* 1 *becomes*

$$\sum_{i=0}^{n}\binom{i}{k} = \sum_{i=k}^{n}\binom{i}{k} = \binom{n+1}{k+1}.$$

Corollary 2. *The shifted binomial sum can be calculated as follows*

$$\sum_{i=m}^{n+m}\binom{i}{k} = \sum_{i=0}^{n}\binom{i+m}{k} = \binom{n+m+1}{k+1} - \binom{m}{k+1}.$$

Proof. Applying Theorem 1 for $q = m$ and replacing n by $n + m$, we obtain the corollary. $\square$

2.2. Repeated binomial sum

Using Theorem 1, we develop a formula for the repeated sum of binomial coefficients.

Theorem 2. *For any $m \in \mathbb{N}^*$ and for any $k, q, n \in \mathbb{N}$ such that $n \geq q$, we have*

$$\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2}\binom{N_1}{k} = \binom{n+m}{k+m} - \sum_{j=1}^{m}\binom{(q-1)+j}{k+j}\binom{(n-q)+(m-j)}{m-j}$$

$$= \binom{n+m}{k+m} - \sum_{j=1}^{m}\binom{(q-1)+j}{(q-1)-k}\binom{(n-q)+(m-j)}{n-q}.$$

Proof. Base case: Verify that the statement is true for $m = 1$.

$$\binom{n+1}{k+1} - \sum_{j=1}^{1}\binom{(q-1)+j}{k+j}\binom{(n-q)+(1-j)}{1-j} = \binom{n+1}{k+1} - \binom{q}{k+1}.$$

From Theorem 1, this case is proven.

Induction hypothesis: Assume the statement is true until m, i.e.,

$$\sum_{N_m=q}^{n}\cdots\sum_{N_1=q}^{N_2}\binom{N_1}{k} = \binom{n+m}{k+m} - \sum_{j=1}^{m}\binom{(q-1)+j}{k+j}\binom{(n-q)+(m-j)}{m-j}.$$

Induction step: We will show that this statement is true for $(m+1)$.

We have to show the following statement to be true:

$$\sum_{N_{m+1}=q}^{n}\cdots\sum_{N_1=q}^{N_2}\binom{N_1}{k} = \binom{n+m+1}{k+m+1} - \sum_{j=1}^{m+1}\binom{(q-1)+j}{k+j}\binom{(n-q)+(m+1-j)}{m+1-j}.$$

$$\sum_{N_{m+1}=q}^{n} \cdots \sum_{N_1=q}^{N_2} \binom{N_1}{k} = \sum_{N_{m+1}=q}^{n} \left(\sum_{N_m=q}^{N_{m+1}} \cdots \sum_{N_1=q}^{N_2} \binom{N_1}{k} \right)$$

$$= \sum_{N_{m+1}=q}^{n} \binom{N_{m+1}+m}{k+m} - \sum_{N_{m+1}=q}^{n} \sum_{j=1}^{m} \binom{(q-1)+j}{k+j}\binom{(N_{m+1}-q)+(m-j)}{m-j}$$

$$= \sum_{N_{m+1}=q}^{n} \binom{N_{m+1}+m}{k+m} - \sum_{j=1}^{m} \sum_{N_{m+1}=q}^{n} \binom{(q-1)+j}{k+j}\binom{(N_{m+1}-q)+(m-j)}{m-j}$$

$$= \sum_{N_{m+1}=q+m}^{n+m} \binom{N_{m+1}}{k+m} - \sum_{j=1}^{m} \binom{(q-1)+j}{k+j} \sum_{N_{m+1}=m-j}^{(n-q)+(m-j)} \binom{N_{m+1}}{m-j}.$$

By applying Theorem 1,

$$\sum_{N_{m+1}=q}^{n} \cdots \sum_{N_1=q}^{N_2} \binom{N_1}{k} = \binom{n+m+1}{k+m+1} - \binom{q+m}{k+m+1} - \sum_{j=1}^{m} \binom{(q-1)+j}{k+j}\binom{(n-q)+(m+1-j)}{m+1-j}.$$

Noticing that

$$\sum_{j=m+1}^{m+1} \binom{(q-1)+j}{k+j}\binom{(n-q)+(m+1-j)}{m+1-j} = \binom{q+m}{k+m+1},$$

hence, the case for $(n+1)$ is proven and the theorem is proven by induction.   □

An important particular case of Theorem 2 is illustrated by the following corollary. In fact, the formula for the repeated sum of binomial coefficients is heavily simplified if the sums are started at 0.

Corollary 3. *For any $m \in \mathbb{N}^*$ and for any $k, n \in \mathbb{N}$, we have*

$$\sum_{N_m=0}^{n} \cdots \sum_{N_1=0}^{N_2} \binom{N_1}{k} = \binom{n+m}{k+m}.$$

Proof. From Theorem 2 for $q = 0$,

$$\sum_{N_m=0}^{n} \cdots \sum_{N_1=0}^{N_2} \binom{N_1}{k} = \binom{n+m}{k+m} - \sum_{j=1}^{m} \binom{j-1}{k+j}\binom{n+(m-j)}{m-j}.$$

We can notice that because $k \geq 0$, then $k+j \geq j > j-1$, thus, $\forall j \in \mathbb{N}$, $\binom{j-1}{k+j} = 0$. Hence, we obtain the corollary.   □

The formula for repeated sums of ones, which is needed to prove the new definition for binomial coefficients, is another crucial particular case of Theorem 2.

Corollary 4. *For any $m \in \mathbb{N}^*$ and for any $q, n \in \mathbb{N}$ such that $n \geq q$, we have*

$$\sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} 1 = \binom{n-q+m}{m}.$$

Proof.

$$\sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} 1 = \sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=0}^{N_2-q} 1 = \sum_{N_m=q}^{n} \cdots \sum_{N_2=0}^{N_3-q} \sum_{N_1=0}^{N_2} 1 = \cdots = \sum_{N_m=0}^{n-q} \cdots \sum_{N_2=0}^{N_3} \sum_{N_1=0}^{N_2} 1.$$

From Corollary 3 for $k = 0$, we obtain the corollary.   □

The sum of a special product of binomial coefficients can also be derived from Theorem 2.

Corollary 5. *For any $m \in \mathbb{N}^*$ and for any $q, n \in \mathbb{N}$ such that $n \geq q$, we have*

$$\sum_{j=1}^{m} \binom{(q-1)+j}{j}\binom{(n-q)+(m-j)}{m-j} = \sum_{j=1}^{m} \binom{(q-1)+j}{q-1}\binom{(n-q)+(m-j)}{n-q} = \binom{n+m}{m} - \binom{n-q+m}{m}.$$

Proof. By applying Theorem 2 for $k = 0$,

$$\sum_{N_m=q}^{n} \cdots \sum_{N_1=q}^{N_2} 1 = \binom{n+m}{m} - \sum_{j=1}^{m} \binom{(q-1)+j}{j}\binom{(n-q)+(m-j)}{m-j}$$

$$= \binom{n+m}{m} - \sum_{j=1}^{m} \binom{(q-1)+j}{q-1}\binom{(n-q)+(m-j)}{n-q}.$$

By comparing with the expression given by Corollary 4, we get this corollary.　　□

3. Binomial coefficients as repeated sums and repeated sequences

In this section, we use the formulas presented in the previous section in order to introduce a new definition of binomial coefficients. First, we prove that a binomial coefficient can be defined as a repeated sum of '1's. Second, we prove that a binomial coefficient can be defined as a repeated sequence.

3.1. Binomial coefficients as repeated sums of ones

In this section, we prove that binomial coefficients can be defined as repeated sums of ones.
A repeated sum of '1's can be expressed as a binomial coefficient as is shown by Theorem 3.

Theorem 3. *A repeated sum of 1s corresponds to the following binomial coefficient,*

$$\sum_{N_m=1}^{n} \cdots \sum_{N_1=1}^{N_2} 1 = \binom{n+m-1}{m}.$$

$$\sum_{N_m=0}^{n} \cdots \sum_{N_1=0}^{N_2} 1 = \binom{n+m}{m}.$$

Proof. The first equality of this theorem is obtained by applying Corollary 4 for $q = 1$ and the second equality is obtained by applying Corollary 4 for $q = 0$.　　□

Remark 3. We can highlight the combinatorical nature of repeated sums of 1s as $\binom{n-1+m}{m}$ represents the number of ways of choosing m elements from n with replacement but without regard to order ([13] pp. 24-25).

A binomial coefficient can be defined as a repeated sum of 1s as shown by Theorem 4.

Theorem 4. *A binomial coefficient corresponds to the following repeated sums of 1s,*

$$\binom{n}{k} = \sum_{N_k=1}^{n-k+1} \cdots \sum_{N_1=1}^{N_2} 1 = \sum_{N_k=0}^{n-k} \cdots \sum_{N_1=0}^{N_2} 1.$$

Proof. The first part of this theorem is obtained by replacing n by $n - k + 1$ in the first equation of Theorem 3. The second part is obtained by replacing n by $n - k$ in the second equation of Theorem 3.　　□

3.2. Binomial coefficients as repeated sequences

In this section, we will go through a series of lemmas in order to prove that any binomial coefficient can be defined as a repeated sequence (defined below). In order to do so, we will first prove that repeated sequences can be explicitly expressed as a repeated sum of ones. Then, using the fact that any repeated sum of ones can be defined as a binomial coefficient (as proven in the previous section), we define these repeated sequences as binomial coefficients.

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3.2.1. **Notation and definitions**

Definition 1. We define the first difference of a sequence (x_n), denoted by Δx_n, as the difference between two consecutive terms of the sequence (x_n):

$$\Delta x_n = x_n - x_{n-1}.$$

Definition 2. Let $m \in \mathbb{N}$. We define the m-th difference of a sequence (x_n), denoted by $\Delta^m x_n$, by the following recurrent relation:

$$\begin{cases} \Delta^m x_n = \Delta^{m-1} x_n - \Delta^{m-1} x_{n-1}, \\ \forall i \in \mathbb{Z}, \Delta^0 x_i = x_i. \end{cases}$$

Definition 3. We define a repeated sequence of degree m, denoted by $\{x_{n,m}\}_{n=1}^{\infty}$, as a sequence whose m-th difference is a constant ($\forall i, j \in \mathbb{N}^*$, $\Delta^m x_{i,m} = \Delta^m x_{j,m}$). $x_{n,m}$ is the n-th term of this repeated sequence of order m.

Remark 4. In order to keep the m-th difference always valid we define an extension of the sequence such that $\forall i \leq 0, x_{i,m} = 0$. However, the properties of the sequence do not extend to this extension. Note that the theorems develop in this section will only apply to repeated sequences which validate the condition imposed by this extension.

Remark 5. For a given order m and a given m-th difference, the repeated sequence is unique. The repeated sequence of order m having $\Delta^m x_{n,m} = \delta \in \mathbb{C}$ is denoted by $\{x_{n,m}^{(\delta)}\}_{n=1}^{\infty}$.

Definition 4. We define the unity repeated sequence, denoted by $\{\hat{x}_{n,m}\}_{n=1}^{\infty}$, as the sequence whose m-th difference is 1 ($\forall i \in \mathbb{N}^*$, $\Delta^m \hat{x}_{i,m} = 1$). $\hat{x}_{n,m}$ is the n-th term of the unity repeated sequence of order m.

Example 1. The unity repeated sequence of order 0, $(\hat{x}_{n,0})$, is the following:

$$\hat{x}_{1,0} = 1, \ \hat{x}_{2,0} = 1, \ \hat{x}_{3,0} = 1, \cdots.$$

Example 2. The unity repeated sequence of order 1, $(\hat{x}_{n,1})$, is the following:

$$\hat{x}_{1,1} = 1, \ \hat{x}_{2,1} = 2 (= 1 + 1), \ \hat{x}_{3,1} = 3 (= 1 + 1 + 1), \cdots.$$

Example 3. The unity repeated sequence of order 2, $(\hat{x}_{n,2})$, is the following:

$$\hat{x}_{1,2} = 1, \ \hat{x}_{2,2} = 3 (= 1 + (1 + 1)), \ \hat{x}_{3,2} = 6 (= 1 + (1 + 1) + (1 + 1 + 1)), \cdots.$$

3.2.2. **Expressions for the m-th difference of a sequence**

In this section, using the recurrent definition of the m-th difference, we develop an additional definition for the m-th difference of a sequence in terms of a lower order difference of this sequence.

Theorem 5. *For any $m, n, p \in \mathbb{N}$ such that $p \leq m$ and for any sequence x_n, we have that*

$$\Delta^m x_n = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p} x_{n-j}.$$

Proof. Base case: Verify true for $m = 0$.

Knowing that $0 \leq p \leq m$ and $m = 0$, hence, $p = 0$.

$$\sum_{j=0}^{0} (-1)^j \binom{0}{j} \Delta^0 x_{n-j} = \Delta^0 x_n.$$

Induction hypothesis: Assume the statement is true until m, i.e.,

$$\Delta^m x_n = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p} x_{n-j}.$$

Induction step: We will show that this statement is true for $(m+1)$.

We have to show the following statement to be true:

$$\Delta^{m+1} x_n = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p+1} x_{n-j}.$$

Knowing that $\Delta^{m+1} x_n = \Delta^m x_n - \Delta^m x_{n-1}$ and applying the induction hypothesis, we get

$$\Delta^{m+1} x_n = \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p} x_{n-j} - \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p} x_{n-j-1}$$

$$= \sum_{j=0}^{p} (-1)^j \binom{p}{j} (\Delta^{m-p} x_{n-j} - \Delta^{m-p} x_{n-j-1})$$

$$= \sum_{j=0}^{p} (-1)^j \binom{p}{j} \Delta^{m-p+1} x_{n-j}.$$

$\square$

Remark 6. As we can see, the binomial coefficient $\binom{p}{j}$ can be defined as the coefficient of $(-1)^j \Delta^{m-p} x_{n-j}$ in the expression of the m-th difference of the sequence (x_n).

In particular, the m-th difference of a sequence can be expressed in terms of its elements.

Corollary 6. *For $p = m$, Theorem 5 becomes*

$$\Delta^m x_n = \sum_{j=0}^{m} (-1)^j \binom{m}{j} x_{n-j}.$$

3.2.3. **Explicit expression for a repeated sequence as a repeated sum of ones**

We are interested in the calculation of the sum of such sequences as well as in the definition of such sequences in terms of binomial coefficients. To do so, in this section, we convert the definition of a repeated sequence of degree m which is in terms of the m-th difference into a definition in terms of a repeated sum of ones. Then, in the later sections, the theorems for repeated sums of ones are used to prove the proposed definition as well as to develop a formula for the repeated sum of a repeated sequence of degree m.

We start by proving the following set of lemmas.

Lemma 1. *For repeated sequences of any order k, the difference of any order m is zero for all terms with an index $i \leq 0$,*

$$\forall k \in \mathbb{N}, \forall m \in \mathbb{N}, \forall i \leq 0, \Delta^m x_{i,k}^{(\delta)} = 0.$$

Proof. Base case: Verify true for $m = 0$.

By definition, $x_{i,k}^{(\delta)} = 0$ for $i \leq 0$. Hence,

$$\forall k \in \mathbb{N}, \forall i \leq 0, \Delta^0 x_{i,k}^{(\delta)} = x_{i,k}^{(\delta)} = 0.$$

Induction hypothesis: Assume the statement is true until m, i.e.,

$$\forall k \in \mathbb{N}, \forall i \leq 0, \Delta^m x_{i,k}^{(\delta)} = 0.$$

Induction step: We will show that this statement is true for $(m + 1)$. We have to show the following statement to be true:

$$\forall k \in \mathbb{N}, \forall i \leq 0, \Delta^{m+1} x_{i,k}^{(\delta)} = 0.$$

By definition,

$$\Delta^{m+1} x_{i,k}^{(\delta)} = \Delta^m x_{i,k}^{(\delta)} - \Delta^m x_{i-1,k}^{(\delta)}.$$

From the induction hypothesis, we get that $\Delta^m x_{i,k}^{(\delta)} = 0$ and $\Delta^m x_{i-1,k}^{(\delta)} = 0$ because, respectively, $i \leq 0$ and $i - 1 \leq -1 \leq 0$. Hence, $\Delta^{m+1} x_{i,k}^{(\delta)} = 0$. $\qquad \square$

Lemma 2. *For repeated sequences of any order k, the difference of any order m of the first term of the sequence is equal to this initial term,*

$$\forall k \in \mathbb{N}, \forall m \in \mathbb{N}, \Delta^m x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}.$$

Proof. Base case: Verify true for $m = 0$.

$$\forall k \in \mathbb{N}, \Delta^0 x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}.$$

Induction hypothesis: Assume the statement is true until m, i.e.,

$$\forall k \in \mathbb{N}, \Delta^m x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}.$$

Induction step: We will show that this statement is true for $(m + 1)$.
   We have to show the following statement to be true:

$$\forall k \in \mathbb{N}, \Delta^{m+1} x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}.$$

By definition,

$$\Delta^{m+1} x_{1,k}^{(\delta)} = \Delta^m x_{1,k}^{(\delta)} - \Delta^m x_{0,k}^{(\delta)}.$$

From Lemma 1, $\Delta^m x_{0,k}^{(\delta)} = 0$. Hence,

$$\Delta^{m+1} x_{1,k}^{(\delta)} = \Delta^m x_{1,k}^{(\delta)}.$$

By applying the induction hypothesis, we get $\Delta^{m+1} x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}$. $\qquad \square$

Remark 7. From Lemma 2 for $k = m$, we see that the first term of a repeated sequence of order m is equal to its m-th difference ($x_{1,m}^{(\delta)} = \Delta^m x_{1,m}^{(\delta)} = \delta$).

Lemma 3. *For any $m, k, n \in \mathbb{N}$ where $n \geq 1$, we have that*

$$\sum_{N=1}^{n} \Delta^m x_{N,k}^{(\delta)} = \Delta^{m-1} x_{n,k}^{(\delta)}.$$

Proof. Base case: Verify true for $n = 1$.
   From Lemma 2, $\Delta^m x_{1,k}^{(\delta)} = \Delta^{m-1} x_{1,k}^{(\delta)} = x_{1,k}^{(\delta)}$,

$$\sum_{N=1}^{1} \Delta^m x_{N,k}^{(\delta)} = \Delta^m x_{1,k}^{(\delta)} = \Delta^{m-1} x_{1,k}^{(\delta)}.$$

Induction hypothesis: Assume the statement is true until n, i.e.,

$$\sum_{N=1}^{n} \Delta^m x_{N,k}^{(\delta)} = \Delta^{m-1} x_{n,k}^{(\delta)}.$$

Induction step: We will show that this statement is true for $(n + 1)$.

We have to show the following statement to be true:

$$\sum_{N=1}^{n+1} \Delta^m x_{N,k}^{(\delta)} = \Delta^{m-1} x_{n+1,k}^{(\delta)}.$$

$$\sum_{N=1}^{n+1} \Delta^m x_{N,k}^{(\delta)} = \sum_{N=1}^{n} \Delta^m x_{N,k}^{(\delta)} + \Delta^m x_{n+1,k}^{(\delta)}.$$

Applying the induction hypothesis to the first term and using the definition of the second term,

$$\sum_{N=1}^{n+1} \Delta^m x_{N,k}^{(\delta)} = \Delta^{m-1} x_{n,k}^{(\delta)} + \left[\Delta^{m-1} x_{n+1,k}^{(\delta)} - \Delta^{m-1} x_{n,k}^{(\delta)} \right] = \Delta^{m-1} x_{n+1,k}^{(\delta)}.$$

$\square$

Lemma 4. *For any $m, k, j, n \in \mathbb{N}$ such that $n \geq 1$ and $1 \leq j \leq m$, we have that*

$$\sum_{N_j=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,k}^{(\delta)} = \Delta^{m-j} x_{n,k}^{(\delta)}.$$

Proof. Base case: Verify true for $j = 1$.

It is straight forward from Lemma 3.

Induction hypothesis: Assume the statement is true until j, i.e.,

$$\sum_{N_j=1}^{n} \cdots \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,k}^{(\delta)} = \Delta^{m-j} x_{n,k}^{(\delta)}.$$

Induction step: We will show that this statement is true for $(j+1)$.

We have to show the following statement to be true:

$$\sum_{N_{j+1}=1}^{n} \cdots \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,k}^{(\delta)} = \Delta^{m-j-1} x_{n,k}^{(\delta)}.$$

Using the induction hypothesis,

$$\sum_{N_{j+1}=1}^{n} \cdots \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,k}^{(\delta)} = \sum_{N_{j+1}=1}^{n} \sum_{N_j=1}^{N_{j+1}} \cdots \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,k}^{(\delta)} = \sum_{N_{j+1}=1}^{n} \Delta^{m-j} x_{N_{j+1},k}^{(\delta)}.$$

Applying Lemma 3, we get the result. $\square$

A particular case of Lemma 4 for $j = k = m$ is as follows. This theorem relates the repeated sequence to the repeated sum of ones.

Theorem 6. *Any element of the m-th degree repeated sequence can be defined as an m-th order repeated sum of ones,*

$$x_{n,m}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)} \sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} 1 = \delta \sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} 1.$$

Proof. A particular case of Lemma 4 for $j = k = m$ is as follows,

$$\sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \Delta^m x_{N_1,m}^{(\delta)} = x_{n,m}^{(\delta)}.$$

By definition of a repeated sequence of degree m, $\Delta^m x_{n,m}^{(\delta)}$ is a constant. Hence, we can take it out of the summation. Doing so gives the desired theorem. $\square$

For repeated unity sequences, Theorem 6 shows that these sequences are equal to repeated sums of ones.

Corollary 7. *Any element of the m-th degree unity repeated sequence can be defined as an m-th order repeated sum of ones,*

$$\hat{x}_{n,m} = \sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} 1.$$

Proof. Knowing that $\Delta^m \hat{x}_{n,m} = 1$, this corollary is a direct consequence of Theorem 6. $\square$

3.2.4. **Explicit expression for a repeated sequence as a binomial coefficient**

We now prove that repeated sequences can be defined as binomial coefficients.

Theorem 7. *For any $m, n \in \mathbb{N}$ where $n \geq 1$, we have*

$$x_{n,m}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)} \binom{n+m-1}{m} = \delta \binom{n+m-1}{m}.$$

Proof. By applying Corollary 3 to Theorem 6, we obtain the result. $\square$

Corollary 8. *For any $m, n \in \mathbb{N}$ where $n \geq 1$, we have*

$$\hat{x}_{n,m} = \binom{n+m-1}{m}.$$

3.2.5. **Sum of repeated sequences**

Now that the needed expressions for repeated sequences were developed, one can proceed to develop formulae for the sum of the terms of such sequences.

Theorem 8. *For any $k \in \mathbb{N}^*$ and for any $m, n \in \mathbb{N}$ where $n \geq 1$, we have*

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} x_{N_1,m}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)} \sum_{N_{k+m}=1}^{n} \cdots \sum_{N_1=1}^{N_2} 1 = \Delta^m x_{n,m}^{(\delta)} \binom{n+m+k-1}{m+k} = x_{n,m+k}^{(\delta)}.$$

Proof. Applying Theorem 6, we get

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} x_{N_1,m}^{(\delta)} = \sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} \sum_{j_m=1}^{N_1} \cdots \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} \Delta^m x_{N_1,m}^{(\delta)}.$$

Knowing that $\forall i, j \in \mathbb{N}^*$, $\Delta^m x_{i,m}^{(\delta)} = \Delta^m x_{j,m}^{(\delta)}$, hence, $\Delta^m x_{N_1,m}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)}$.

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} x_{N_1,m}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)} \sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} \sum_{j_m=1}^{N_1} \cdots \sum_{j_2=1}^{j_3} \sum_{j_1=1}^{j_2} 1 = \Delta^m x_{n,m}^{(\delta)} \sum_{N_{k+m}=1}^{n} \cdots \sum_{N_1=1}^{N_2} 1.$$

The first equality is proven. The second equality is obtained by applying Theorem 1.

Knowing that $\Delta^{m+k} x_{n,m+k}^{(\delta)} = \Delta^m x_{n,m}^{(\delta)} = \delta$, we apply Theorem 6 to get

$$x_{n,m+k}^{(\delta)} = \Delta^m x_{n,m+k}^{(\delta)} \sum_{N_{m+k}=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} 1 = \Delta^m x_{n,m}^{(\delta)} \sum_{N_{m+k}=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} 1.$$

This completes the proof of the third equality. $\square$

Corollary 9. *For any $k \in \mathbb{N}^*$ and for any $m, n \in \mathbb{N}$ where $n \geq 1$, we have*

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} \hat{x}_{N_1,m} = \sum_{N_{k+m}=1}^{n} \cdots \sum_{N_1=1}^{N_2} 1 = \binom{n+m+k-1}{m+k} = \hat{x}_{n,m+k}.$$

Proof. Applying Theorem 8 and noting that $\Delta^m \hat{x}_{n,m} = 1$, we obtain this result.   □

Example 4. The sequence $(\hat{x}_{n,1})$ is a sequence of terms where the difference of two consecutive terms is 1: $1, 2, 3, 4, 5, \ldots$. Using Corollary 9, we calculate the repeated sum of order 3 of its first 10 terms as follows:

$$\sum_{N_3=1}^{10} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \hat{x}_{N_1,1} = \binom{10+1+3-1}{1+3} = \binom{13}{4} = 715.$$

Example 5. The sequence $(\hat{x}_{n,2})$ is a sequence of terms where the difference of two consecutive terms is increasing by 1: $1, 3, 6, 10, 15, \ldots$. Using Corollary 9, we calculate the repeated sum of order 3 of its first 10 terms as follows:

$$\sum_{N_3=1}^{10} \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \hat{x}_{N_1,2} = \binom{10+2+3-1}{2+3} = \binom{14}{5} = 2002.$$

4. Reduction of Binomial-Harmonic sums

The harmonic sum has been studied independently by Oresme [14], Mengoli [15], Johann Bernoulli [16], Jacob Bernoulli [17,18], and most importantly by Euler. A particularly interesting variant of this sum is that combining binomial coefficients with harmonic sums. We will refer to such sums as *binomial-harmonic sums*. In this section, we develop various formulas for the reduction and calculation of such sums. We also present a formula relating the repeated harmonic sum to the shifted harmonic sum.

4.1. Binomial-Harmonic sum

We begin by proving the following reduction formula to simplify Binomial-Harmonic sums.

Theorem 9. *For any $m, p, n \in \mathbb{N}$ such that $n \geq p$, we have*

$$\sum_{N=p}^{n} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+1}{m+1} \sum_{i=2+m}^{n+m+1} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

Proof. Base case: Verify true for $n = p$.

$$\binom{p+m+1}{m+1} \sum_{i=2+m}^{p+m+1} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i} = \binom{p+m+1}{m+1} \frac{1}{p+m+1} + \left[\binom{p+m+1}{m+1} - \binom{p+m}{m+1} \right] \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

From the Pascal's Triangle identity, $\binom{p+m+1}{m+1} - \binom{p+m}{m+1} = \binom{p+m}{m}$. In addiction,

$$\binom{p+m+1}{m+1} \frac{1}{p+m+1} = \binom{p+m}{m} \frac{1}{m+1}.$$

Hence, substituting back, we get

$$\binom{p+m+1}{m+1} \sum_{i=2+m}^{p+m+1} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i} = \binom{p+m}{m} \sum_{i=1+m}^{p+m} \frac{1}{i} = \sum_{N=p}^{p} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i}.$$

Induction hypothesis: Assume the statement is true until n, i.e.,

$$\sum_{N=p}^{n} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+1}{m+1} \sum_{i=2+m}^{n+m+1} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

Induction step: We will show that this statement is true for $(n+1)$. We have to show the following statement to be true:

$$\sum_{N=p}^{n+1} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+2}{m+1} \sum_{i=2+m}^{n+m+2} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

$$\sum_{N=p}^{n+1} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \sum_{N=p}^{n} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} + \binom{n+m+1}{m} \sum_{i=1+m}^{n+m+1} \frac{1}{i}.$$

From the induction hypothesis,

$$\sum_{N=p}^{n+1} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+1}{m+1} \sum_{i=2+m}^{n+m+1} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i} + \binom{n+m+1}{m} \sum_{i=1+m}^{n+m+1} \frac{1}{i}$$

$$= \left[\binom{n+m+1}{m+1} + \binom{n+m+1}{m} \right] \sum_{i=2+m}^{n+m+1} \frac{1}{i} + \binom{n+m+1}{m} \frac{1}{m+1} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

From the Pascal's Triangle identity, $\binom{n+m+1}{m+1} + \binom{n+m+1}{m} = \binom{n+m+2}{m+1}$. In addition,

$$\binom{n+m+1}{m} \frac{1}{m+1} = \frac{(n+m+1)!}{m!(n+1)!} \frac{1}{m+1} = \frac{(n+m+2)!}{(m+1)!(n+1)!} \frac{1}{n+m+2} = \binom{n+m+2}{m+1} \frac{1}{n+m+2}.$$

Hence,

$$\sum_{N=p}^{n+1} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+2}{m+1} \sum_{i=2+m}^{n+m+2} \frac{1}{i} - \binom{p+m}{m+1} \sum_{i=2+m}^{p+m} \frac{1}{i}.$$

$\square$

Corollary 10. *For $p = 1$, Theorem 9 becomes*

$$\sum_{N=1}^{n} \binom{N+m}{m} \sum_{i=1+m}^{N+m} \frac{1}{i} = \binom{n+m+1}{m+1} \sum_{i=2+m}^{n+m+1} \frac{1}{i}.$$

4.2. Repeated Binomial-Harmonic sum

Now we generalize Theorem 9 and develop a formula for simplifying repeated Binomial-Harmonic sums.

Theorem 10. *For any $k \in \mathbb{N}^*$ and for any $m, p, n \in \mathbb{N}$ such that $n \geq p$, we have*

$$\sum_{N_k=p}^{n} \cdots \sum_{N_1=p}^{N_2} \left[\binom{N_1+m}{m} \sum_{i=1+m}^{N_1+m} \frac{1}{i} \right] = \binom{n+m+k}{m+k} \sum_{i=1+m+k}^{n+m+k} \frac{1}{i} - \sum_{j=0}^{k-1} \binom{n-p+j}{j} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}.$$

Proof. Base case: Verify true for $k = 1$.

For $k = 1$ this theorem will reduce to Theorem 9. Hence, the case for $k = 1$ is proven.

Induction hypothesis: Assume the statement is true until k, i.e.,

$$\sum_{N_k=p}^{n} \cdots \sum_{N_1=p}^{N_2} \left[\binom{N_1+m}{m} \sum_{i=1+m}^{N_1+m} \frac{1}{i} \right] = \binom{n+m+k}{m+k} \sum_{i=1+m+k}^{n+m+k} \frac{1}{i} - \sum_{j=0}^{k-1} \binom{n-p+j}{j} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}.$$

Induction step: We will show that this statement is true for $(k+1)$.

We have to show the following statement to be true:

$$\sum_{N_{k+1}=p}^{n} \cdots \sum_{N_1=p}^{N_2} \left[\binom{N_1+m}{m} \sum_{i=1+m}^{N_1+m} \frac{1}{i} \right] = \binom{n+m+k+1}{m+k+1} \sum_{i=2+m+k}^{n+m+k+1} \frac{1}{i} - \sum_{j=0}^{k} \binom{n-p+j}{j} \binom{p+m+k-j}{m+k-j+1} \sum_{i=2+m+k-j}^{p+m+k-j} \frac{1}{i}.$$

Let us denote the first term of the equality as B.

$$B = \sum_{N_{k+1}=p}^{n} \left\{ \sum_{N_k=p}^{N_{k+1}} \cdots \sum_{N_1=p}^{N_2} \left[\binom{N_1+m}{m} \sum_{i=1+m}^{N_1+m} \frac{1}{i} \right] \right\}.$$

Applying the induction hypothesis, we get

$$B = \sum_{N_{k+1}=p}^{n} \binom{N_{k+1}+m+k}{m+k} \sum_{i=1+m+k}^{N_{k+1}+m+k} \frac{1}{i} - \sum_{N_{k+1}=p}^{n} \sum_{j=0}^{k-1} \binom{N_{k+1}-p+j}{j} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}$$

$$= \sum_{N_{k+1}=p}^{n} \binom{N_{k+1}+m+k}{m+k} \sum_{i=1+m+k}^{N_{k+1}+m+k} \frac{1}{i} - \sum_{j=0}^{k-1} \left\{ \sum_{N_{k+1}=p}^{n} \binom{N_{k+1}-p+j}{j} \right\} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}.$$

Applying Theorem 9 to the first term and applying Theorem 1 to the term in curly brackets,

$$B = \binom{n+m+k+1}{m+k+1} \sum_{i=2+m+k}^{n+m+k+1} \frac{1}{i} - \binom{p+m+k}{m+k+1} \sum_{i=2+m+k}^{p+m+k} \frac{1}{i} - \sum_{j=0}^{k-1} \binom{n-p+j+1}{j+1} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}$$

$$= \binom{n+m+k+1}{m+k+1} \sum_{i=2+m+k}^{n+m+k+1} \frac{1}{i} - \binom{p+m+k}{m+k+1} \sum_{i=2+m+k}^{p+m+k} \frac{1}{i} - \sum_{j=1}^{k} \binom{n-p+j}{j} \binom{p+m+k-j}{m+k-j+1} \sum_{i=2+m+k-j}^{p+m+k-j} \frac{1}{i}$$

$$= \binom{n+m+k+1}{m+k+1} \sum_{i=2+m+k}^{n+m+k+1} \frac{1}{i} - \sum_{j=0}^{k} \binom{n-p+j+1}{j+1} \binom{p-1+m+k-j}{m+k-j} \sum_{i=1+m+k-j}^{p-1+m+k-j} \frac{1}{i}.$$

Hence, the statement holds true for $(k+1)$. Thus, the result is proved.   □

Corollary 11. *For p = 1, Theorem 10 becomes*

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} \left[\binom{N_1+m}{m} \sum_{i=1+m}^{N_1+m} \frac{1}{i} \right] = \binom{n+m+k}{m+k} \sum_{i=1+m+k}^{n+m+k} \frac{1}{i}.$$

A particular case of Corollary 11 which is of major interest is the repeated harmonic sum.

Corollary 12. *The repeated harmonic sum of order $(m+1)$ is related to the shifted harmonic sum of order 1 by the following relation,*

$$\sum_{N_{m+1}=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \frac{1}{N_1} = \binom{n+m}{m} \sum_{N=1+m}^{n+m} \frac{1}{N}.$$

Proof. Applying Corollary 11 with $m = 0$, we get this result.   □

5. Application to repeated sums

5.1. Reduction of a repeated sum

Let us denote the repeated sum of order m of the sequence a_N with lower and upper bounds respectively q and n as $S_{m,q,n}(a_N)$. For simplicity, we will denote it as $S_{m,q,n}$.

$$S_{m,q,n} = \sum_{N_m=q}^{n} \sum_{N_{m-1}=q}^{N_m} \cdots \sum_{N_1=q}^{N_2} a_{N_1}.$$

Recursively, it can be defined by:

$$\begin{cases} S_{m,q,n} = \sum_{N_m=q}^{n} S_{m-1,q,N_m}, \\ S_{0,q,n} = a_n \quad \forall q,n \in \mathbb{N}. \end{cases}$$

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5.1.1. **Variation Formula**

In this section, we will present formulas to express the variation of a repeated sum of order m ($S_{m,q,n+1} - S_{m,q,n}$) in terms of lower order repeated sums. Equivalently, these formulas can be used to express $S_{m,q,n+1}$ in terms of $S_{m,q,n}$ and lower order repeated sums.

We begin by presenting the variation formula that allows $S_{m,q,n+1}$ to be expressed in terms of $S_{m,q,n}$ and of repeated sums of order going from 0 to $(m-1)$.

Theorem 11. *For any $n, q \in \mathbb{N}$ such that $n \geq q$, and for any $m \in \mathbb{N}^*$, we have*

$$\sum_{N_m=q}^{n+1} \sum_{N_{m-1}=q}^{N_m} \cdots \sum_{N_1=q}^{N_2} a_{N_1} = \sum_{k=1}^{m} \left(\sum_{N_k=q}^{n} \sum_{N_{k-1}=q}^{N_k} \cdots \sum_{N_1=q}^{N_2} a_{N_1} \right) + a_{n+1}.$$

Or using the notation,

$$S_{m,q,n+1} = \sum_{k=1}^{m} S_{k,q,n} + S_{0,q,n+1}.$$

Proof. In [1], the author presented the following variation formula:

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} = \sum_{k=0}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) \left(\sum_{N_k=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(k);N_k} \cdots a_{(1);N_1} \right).$$

Setting $a_{(m);N} = \cdots = a_{(2);N} = 1$ and $a_{(1);N} = a_N$, we obtain this result. $\square$

The variation of a repeated sum can also be expressed in terms of only a certain range of lower order repeated sums. In other words, $S_{m,q,n+1}$ can be expressed in terms of $S_{m,q,n}$ and of repeated sums of order going only from p to $(m-1)$. To do so, we give the following theorem:

Theorem 12. *For any $n, q \in \mathbb{N}$ such that $n \geq q$, and for any $m, p \in \mathbb{N}^*$ such that $p \leq m$, we have*

$$\sum_{N_m=q}^{n+1} \sum_{N_{m-1}=q}^{N_m} \cdots \sum_{N_1=q}^{N_2} a_{N_1} = \sum_{k=p+1}^{m} \left(\sum_{N_k=q}^{n} \sum_{N_{k-1}=q}^{N_k} \cdots \sum_{N_1=q}^{N_2} a_{N_1} \right) + \sum_{N_p=q}^{n+1} \sum_{N_{p-1}=q}^{N_p} \cdots \sum_{N_1=q}^{N_2} a_{N_1}.$$

Or using the notation,

$$S_{m,q,n+1} = \sum_{k=p+1}^{m} S_{k,q,n} + S_{p,q,n+1}.$$

Proof. In [1], the author presented the following variation formula:

$$\sum_{N_m=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(m);N_m} \cdots a_{(1);N_1} = \sum_{k=p+1}^{m} \left(\prod_{j=0}^{m-k-1} a_{(m-j);n+1} \right) \left(\sum_{N_k=q}^{n} \cdots \sum_{N_1=q}^{N_2} a_{(k);N_k} \cdots a_{(1);N_1} \right)$$
$$+ \left(\prod_{j=0}^{m-p-1} a_{(m-j);n+1} \right) \left(\sum_{N_p=q}^{n+1} \cdots \sum_{N_1=q}^{N_2} a_{(p);N_p} \cdots a_{(1);N_1} \right).$$

Setting $a_{(m);N} = \cdots = a_{(2);N} = 1$ and $a_{(1);N} = a_N$, we obtain this result. $\square$

5.1.2. **Reduction Formula**

Now that we have proven the needed binomial coefficient definitions and identities, we derive a formula to simplify the repeated sum of any sequence a_N.

Theorem 13. *For any $m \in \mathbb{N}^*$, for any $q, n \in \mathbb{N}$ such that $n \geq q$ and for any sequence a_N defined in the interval $[q, n]$, we have*

$$\sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} a_{N_1} = \sum_{i=q}^{n} \binom{n+(m-1)-i}{m-1} a_i.$$

Proof. Applying Theorem 3.3 from [1], for $a_{(m);N_m} = \cdots = a_{(2);N_2} = 1$ and $a_{(1);N_1} = a_{N_1}$, we get

$$\sum_{N_m=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} a_{N_1} = \sum_{N_1=q}^{n} a_{N_1} \sum_{N_m=N_1}^{n} \cdots \sum_{N_3=N_1}^{N_4} \sum_{N_2=N_1}^{N_3} 1.$$

By applying Corollary 4 to the $(m-1)$ inner sums, we have

$$\sum_{N_m=N_1}^{n} \cdots \sum_{N_3=N_1}^{N_4} \sum_{N_2=N_1}^{N_3} 1 = \binom{n - N_1 + m - 1}{m - 1}.$$

Hence, substituting back, we get the desired result. □

Theorem 13 offers a more computationally efficient way of calculating repeated sums as computing the repeated sum directly requires adding up $\binom{n-q+m}{m}$ terms while using the simplified form requires adding up only $n - q + 1$ terms. Also notice that the number of terms added in the simplified form is independent of the order m. For example, let us consider a repeated sum of order $m = 10$ and with lower and upper bounds respectively $q = 1$ and $n = 10$: computing this repeated sum directly requires adding 92378 terms while using the theorem requires adding only 10 terms.

Remark 8. From Theorem 13, we can deduce that a term a_i appears $\binom{n+(m-1)-i}{m-1}$ times in the repeated sum of the sequence a_N.

Corollary 13. *If the sums start at* 1, *Theorem* 13 *becomes*

$$\sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} a_{N_1} = \sum_{i=1}^{n} \binom{n + (m - 1) - i}{m - 1} a_i.$$

5.1.3. **Application to harmonic sums and repeated harmonic sums**

In this section, we apply Theorem 13 to develop some formulae for harmonic sums and repeated harmonic sums. First, we introduce a formula for simplifying repeated harmonic sums.

Theorem 14. *For any* $m, n \in \mathbb{N}^*$, *we have*

$$\sum_{N_m=1}^{n} \cdots \sum_{N_2=1}^{N_3} \sum_{N_1=1}^{N_2} \frac{1}{N_1} = \sum_{i=1}^{n} \binom{n + (m - 1) - i}{m - 1} \frac{1}{i} = \binom{n + m - 1}{m - 1} \sum_{i=m}^{n+m-1} \frac{1}{i}.$$

Proof. By equating the expression of the repeated harmonic sum obtained from Corollary 12 with that obtained from Theorem 13 for $a_N = \frac{1}{N}$, we obtain this result. □

Now we present identities related to the harmonic sum. We start with the following identity:

Theorem 15. *For any* $m, n \in \mathbb{N}$ *such that* $n \geq 1$, *we have*

$$\sum_{i=1}^{n} \frac{\binom{n}{i}}{\binom{n+m}{i}} \frac{1}{i} = \sum_{i=1}^{n} \left[\prod_{k=0}^{i-1} \frac{(n - k)}{(n + m - k)} \right] \frac{1}{i} = \sum_{i=1+m}^{n+m} \frac{1}{i}.$$

Proof. From Theorem 14 with m substituted by $(m + 1)$,

$$\sum_{i=1}^{n} \binom{n + m - i}{m} \frac{1}{i} = \binom{n + m}{m} \sum_{i=1+m}^{n+m} \frac{1}{i}.$$

Knowing that

$$\frac{\binom{n+m-i}{m}}{\binom{n+m}{m}} = \frac{\binom{n}{i}}{\binom{n+m}{i}} = \prod_{k=0}^{i-1} \frac{(n - k)}{(n + m - k)},$$

hence, substituting back, we obtain the result. □

In order to prove the second identity, we need to first prove the following two lemmas:

Lemma 5. *For any $k \in \mathbb{N}^*$, we have*

$$\sum_{i=1}^{2k} \frac{1}{i} = (2k+1) \sum_{i=1}^{k} \frac{1}{(2k+1-i)i} = \frac{(2k+1)}{2} \sum_{i=1}^{2k} \frac{1}{(2k+1-i)i}.$$

Proof. We split the sum, modify its parts, then recombine them to obtain,

$$\sum_{i=1}^{2k} \frac{1}{i} = \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=k+1}^{2k} \frac{1}{i} = \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=1}^{k} \frac{1}{2k+1-i} = (2k+1) \sum_{i=1}^{k} \frac{1}{(2k+1-i)i}.$$

Similarly,

$$\frac{(2k+1)}{2} \sum_{i=1}^{2k} \frac{1}{(2k+1-i)i} = \frac{(2k+1)}{2} \left[\sum_{i=1}^{k} \frac{1}{(2k+1-i)i} + \sum_{i=k+1}^{2k} \frac{1}{(2k+1-i)i} \right]$$

$$= \frac{(2k+1)}{2} \left[\sum_{i=1}^{k} \frac{1}{(2k+1-i)i} + \sum_{i=1}^{k} \frac{1}{(2k+1-i)i} \right]$$

$$= (2k+1) \sum_{i=1}^{k} \frac{1}{(2k+1-i)i}.$$

Equating, we obtain the result. □

Lemma 6. *For any $k \in \mathbb{N}^*$, we have*

$$\sum_{i=1}^{2k-1} \frac{1}{i} = (2k) \sum_{i=1}^{k} \frac{1}{(2k-i)i} - \frac{1}{k} = (2k) \sum_{i=1}^{k-1} \frac{1}{(2k-i)i} + \frac{1}{k} = (k) \sum_{i=1}^{2k-1} \frac{1}{(2k-i)i}.$$

Proof. We split the sum, modify its parts, then recombine them to obtain,

$$\sum_{i=1}^{2k-1} \frac{1}{i} = \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=k}^{2k-1} \frac{1}{i} - \frac{1}{k} = \sum_{i=1}^{k} \frac{1}{i} + \sum_{i=1}^{k} \frac{1}{2k-i} - \frac{1}{k} = (2k) \sum_{i=1}^{k} \frac{1}{(2k-i)i} - \frac{1}{k}.$$

Similarly,

$$\sum_{i=1}^{2k-1} \frac{1}{i} = \sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=k+1}^{2k-1} \frac{1}{i} + \frac{1}{k} = \sum_{i=1}^{k-1} \frac{1}{i} + \sum_{i=1}^{k-1} \frac{1}{2k-i} + \frac{1}{k} = (2k) \sum_{i=1}^{k-1} \frac{1}{(2k-i)i} + \frac{1}{k}.$$

Finally, to obtain the last part of this lemma, we perform the following,

$$(k) \sum_{i=1}^{2k-1} \frac{1}{(2k-i)i} = (k) \left[\sum_{i=1}^{k-1} \frac{1}{(2k-i)i} + \frac{1}{k^2} + \sum_{i=k+1}^{2k-1} \frac{1}{(2k-i)i} \right]$$

$$= (k) \left[\sum_{i=1}^{k-1} \frac{1}{(2k-i)i} + \sum_{i=1}^{k-1} \frac{1}{(2k-i)i} \right] + \frac{1}{k}$$

$$= (2k) \sum_{i=1}^{k-1} \frac{1}{(2k-i)i} + \frac{1}{k}.$$

Equating, we obtain the result. □

Now that both needed lemmas have been proven, we give the second identity.

Theorem 16. *For any $n \in \mathbb{N}^*$, we have*

$$\frac{n+1}{2} \sum_{i=1}^{n} \frac{1}{(n+1-i)i} = \sum_{i=1}^{n} \frac{1}{i}.$$

Proof. The proof will be divided into proving two complementary cases: the case when n is even and the case when n is odd.

- For $n = 2k$ (where $k \in \mathbb{N}^*$):
  This case corresponds to Lemma 5 that we have already proven.
- For $n = 2k - 1$ (where $k \in \mathbb{N}^*$):
  This case corresponds to Lemma 6 that we have already proven.

The formula holds true for both even and odd values of n, hence, it holds for any $n \geq 1$. $\qquad\square$

5.2. Reduction of the repeated "Binomial-Sequence" sum

In this section, we prove a generalization of Theorem 13. This generalization is illustrated in the following theorem that simplifies the repeated "Binomial-Sequence" sum.

Theorem 17. *For any $k \in \mathbb{N}^*$, for any $m, q, n \in \mathbb{N}$ where $n \geq q$ and for any sequence a_N defined in the interval $[q, n]$, we have*

$$\sum_{N_k=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} \binom{(N_2 - N_1) + m}{m} a_{N_1} = \sum_{N=q}^{n} \binom{(n - N) + m + k - 1}{m + k - 1} a_N.$$

Proof. By applying Theorem 13 to the inner sum, we have

$$\sum_{N_k=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{N_1=q}^{N_2} \binom{(N_2 - N_1) + m}{m} a_{N_1} = \sum_{N_k=q}^{n} \cdots \sum_{N_2=q}^{N_3} \sum_{j_{m+1}=q}^{N_2} \cdots \sum_{j_1=q}^{j_2} a_{j_1}$$

$$= \sum_{i_{m+k}=q}^{n} \cdots \sum_{i_2=q}^{i_3} \sum_{i_1=q}^{i_2} a_{i_1}$$

$$= \sum_{N=q}^{n} \binom{(n - N) + m + k - 1}{m + k - 1} a_N.$$

$\qquad\square$

Corollary 14. *If the summations start at 1, Theorem 17 becomes,*

$$\sum_{N_k=1}^{n} \cdots \sum_{N_1=1}^{N_2} \binom{(N_2 - N_1) + m}{m} a_{N_1} = \sum_{N=1}^{n} \binom{(n - N) + m + k - 1}{m + k - 1} a_N.$$

For the simple "Binomial-Sequence" sum, Theorem 17 and Corollary 14 reduce to the following:

Theorem 18. *For any $m \in \mathbb{N}^*$, for any $q, n \in \mathbb{N}$ where $n \geq q$ and for any sequence a_N defined in the interval $[q, n]$, we have*

$$\sum_{N_2=q}^{n} \sum_{N_1=q}^{N_2} \binom{(N_2 - N_1) + (m - 1)}{m - 1} a_{N_1} = \sum_{N=q}^{n} \binom{(n - N) + m}{m} a_N.$$

Corollary 15. *If the summations start at 1, Theorem 18 becomes,*

$$\sum_{N_2=1}^{n} \sum_{N_1=1}^{N_2} \binom{(N_2 - N_1) + (m - 1)}{m - 1} a_{N_1} = \sum_{N=1}^{n} \binom{(n - N) + m}{m} a_N.$$

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