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Common Fixed-Point Theorem for Expansive Mappings in Dualistic Partial Metric Spaces

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Authors' contributions

This work was carried out in collaboration among all authors. All authors read and approved the final manuscript.

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Abstract

O'Neill [1] introduces the concept of dualistic partial metric space. In this study, we prove some common fixed-point theorems for dualistic expanding mappings defined on a dualistic partial metric space. Some famous conclusions of [2] and [3] are extended and generalized by our result. Additionally, we offer an example that demonstrates the value of these dualistic expanding mappings.

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1 Introduction

"Because metric fixed-point theory has so many applications in applied mathematics and the sciences, it is becoming more and more important in mathematics. The conventional understanding of a metric space has been generalized in several ways. A partial metric space, which Matthews developed and examined, is one such generalization" [4]. He verified the exact correspondence between the so-called weightable quasi-metric spaces and partial metric spaces. Partial metrics have certain generalizations. One major modification to Matthews' formulation of the partial metric, for instance, was suggested by O'Neill [1] and involved moving it range from $[0,\infty)$ to $(-\infty,\infty)$. A pair $(\mathcal{U},d^*)$ where $\mathcal{U}$ is a nonempty set and d^* is a dualistic partial metric on $\mathcal{U}$ is referred to as a dualistic partial metric space. According to [1], "the partial metrics in the O'Neill sense shall be called dualistic partial metrics. In this way, O'Neill established several links between partial metrics and the topological aspects of domain theory". The study of Banach's contraction principle served as the foundation for contractive conditions. These conditions have been used in many generalized metric spaces for fixed point theorems. After that, expansive conditions were added [3], and expansive mappings were used to provide new fixed-point solutions. For different contractive or expansive mappings, several fixed-point findings have still been studied utilizing the concepts of metric space and partial metric space. View [5,6,2,7,8,9,10] and [11] for additional information.

The existence of fixed points for a particular class of mappings known as "expansive mappings" is addressed by the fixed-point theorem for expansive mappings, a finding in fixed point theory. According to the theorem, there is always at least one fixed point in an expansive mapping defined on a nonempty bounded metric space [12,13]. A concept of non-contraction is reflected by expansive mappings, which lead points to spread apart during the transformation. Put otherwise, the theorem asserts that a mapping can have invariant points even in the absence of a contraction in the distances between its points. The idea behind it is to expand upon the conventional fixed-point theorem for contraction mappings, which asserts that every contraction mapping has a single fixed point [14,15].

Theorem 1.1 Let $\mathcal{U}$ be a bounded metric space that is not empty, and let f be an expansive mapping from $\mathcal{U}$ to $\mathcal{U}$. Then, f has a fixed point.

A number of diverse scenarios have been expanded upon using this theorem [9,16,17]. Applications of the expansive mappings fixed point theorem can be found in dynamical systems, differential equations, and nonlinear analysis. The idea of expansive mappings makes sense in these domains [18,19,20]. Expansive mappings, for instance, are employed in nonlinear analysis to examine the behavior of dynamical systems [21]. Expanding mappings are used in differential and integral equations to examine the stability of solutions [22]. Expansive mappings are widely utilized in dynamical systems to examine the existence of chaotic attractors [21]. This technique is applied in many different domains, such as the study of dynamics in mathematical biology and economics.

In a dualistic partial metric space, the purpose of this study is to show several common fixed-point theorems under different dualistic expansive mappings. Some famous conclusions of [2] and [3] are extended and generalized by our result. Furthermore, we validate our findings using an example.

2 Preliminaries

To make this work self-contained, we review key definitions and foundational concepts in mathematics.

Definition 2.1. (see [4]) Let $\mathcal{U}$ be a non-empty set. A partial metric on $\mathcal{U}$ is a function $d:\mathcal{U}\times\mathcal{U}\to[0,\infty)$ complying with following axioms, for all $\mathrm{x},\mathrm{y},\mathrm{z}\in\mathcal{U}$

(d_1) $\mathrm{x}=\mathrm{y}\Leftrightarrow d(\mathrm{x},\mathrm{y})=d(\mathrm{x},\mathrm{x})=d(\mathrm{y},\mathrm{y})$,
(d_2) $d(\mathrm{x},\mathrm{x})\le d(\mathrm{x},\mathrm{y})$,

(d_3) $d(x,y) = d(y,x)$,
(d_4) $d(x,y) \leq d(x,z) + d(z,y) - d(z,z)$.

The pair $(\mathcal{U}, d)$ is called a partial metric space.

Definition 2.2. (see [1]) Let $\mathcal{U}$ be a non-empty set. A dualistic partial metric on $\mathcal{U}$ is a function $d^*: \mathcal{U} \times \mathcal{U} \to (-\infty, \infty)$ satisfying the following axioms, for all $x, y, z \in \mathcal{U}$

(d_1^*) $x = y \Leftrightarrow d^*(x,y) = d^*(x,x) = d^*(y,y)$,
(d_2^*) $d^*(x,x) \leq d^*(x,y)$,
(d_3^*) $d^*(x,y) = d^*(y,x)$,
(d_4^*) $d^*(x,z) + d^*(y,y) \leq d^*(x,y) + d^*(y,z)$.

The pair $(\mathcal{U}, d^*)$ is called a dualistic partial metric space.

Remark 2.3 Noting that each partial metric is a dualistic partial metric, but the converse is false. Indeed, define d^* on $(-\infty, \infty)$ as $d^*(x,y) = \max\{x,y\}, \forall x, y \in (-\infty, \infty)$. Obviously, d^* is a dualistic partial metric on $(-\infty, \infty)$. Since $d^*(x,y) < 0 \notin [0, \infty), \forall\, x, y \in (-\infty, 0)$ and then d^* is not a partial metric on $(-\infty, \infty)$. This confirms our remark.

Example 2.4 (see [23,1])

(1) Define $d_\rho^*: \mathcal{U} \times \mathcal{U} \to (-\infty, \infty)$ by $d_\rho^*(x,y) = \rho(x,y) + b$, where ρ is a metric on a nonempty set $\mathcal{U}$ and $b \in (-\infty, \infty)$ is arbitrary constant, then it is easy to check that d_ρ^* verifies axioms $(d_1^*) - (d_4^*)$ and hence $(\mathcal{U}, d^*)$ is a dualistic partial metric space.
(2) Let d be a partial metric defined on a non-empty set $\mathcal{U}$. The function $d^*: \mathcal{U} \times \mathcal{U} \to (-\infty, \infty)$ defined by $d^*(x,y) = d(x,y) - d(x,x) - d(y,y)$ satisfies the axioms $(d_1^*) - (d_4^*)$ and so it defines a dualistic partial metric on $\mathcal{U}$. Note that $d^*(x,y)$ may have negative values.
(3) Let $\mathcal{U} = (-\infty, \infty)$. Define $d^*: \mathcal{U} \times \mathcal{U} \to (-\infty, \infty)$ by $d^*(x,y) = |x - y|$ if $x \neq y$ and $d^*(x,y) = -\beta$ if $x = y$ and $\beta > 0$. We can easily see that d^* is a dualistic partial metric on $\mathcal{U}$.

O'Neill [1] established that each dualistic partial metric d^* on $\mathcal{U}$ generates a T_0 topology $\tau(d^*)$ on $\mathcal{U}$ having a base, the family of d^*-balls $\{\mathcal{B}_{d^*}(x,\epsilon) \mid x \in \mathcal{U}, \epsilon > 0\}$, where

$$\mathcal{B}_{d^*}(x,\epsilon) = \{y \in \mathcal{U} \mid d^*(x,y) < d^*(x,x) + \epsilon\}. \tag{2.1}$$

If $(\mathcal{U}, d^*)$ is a dualistic partial metric space, then the function $\rho_{d^*}: \mathcal{U} \times \mathcal{U} \to [0, \infty)$ defined by

$$\rho_{d^*}(x,y) = d^*(x,y) - d^*(x,x), \tag{2.2}$$

defines a quasi-metric on $\mathcal{A}$ such that $\tau(d^*) = \tau(\rho_{d^*})$ and

$$\rho_{d^*}^m(x,y) = \max\{\rho_{d^*}(x,y), \rho_{d^*}(y,x)\}, \tag{2.3}$$

defines a metric on $\mathcal{U}$.

Definition 2.5 (see [23]) Let $(\mathcal{U}, d^*)$ be a dualistic partial metric space.

1. A sequence $\{x_n\}$ in $\mathcal{U}$ is said to converge or to be convergent if there is a $x \in \mathcal{U}$ such that $\lim_{n\to\infty} d^*(x_n, x) = d^*(x,x)$. x is called the limit of $\{x_n\}$ and we write $x_n \to x$.
2. A sequence $\{x_n\}$ in $\mathcal{U}$ is said to be Cauchy sequence if $\lim_{n,m\to\infty} d^*(x_n, x_m)$ exists and is finite.
3. A dualistic partial metric space $\mathcal{U} = (\mathcal{U}, d^*)$ is said to be complete if every Cauchy sequence $\{x_n\}$ in $\mathcal{U}$ converges, with respect to $\tau(d^*)$, to a point $x \in \mathcal{U}$ such that $d^*(x,x) = \lim_{n,m\to\infty} d^*(x_n, x_m)$.

Remark 2.6 For a sequence, convergence with respect to metric space may not imply convergence with respect to dualistic partial metric space. Indeed, if we take $\beta = 1$ and $\left\{x_n = \frac{1-n}{n} : n \geq 1\right\}_{n\in\mathbb{N}} \subset \mathcal{U}$ as in Example 2.4(3). Mention that $\lim_{n\to\infty} \rho(x_n, -1) = -1$ and therefore, $x_n \to -1$ with respect to ρ. On the other hand, we make a conclusion that $x_n \nrightarrow -1$ with respect to d^* because

$$\lim_{n\to\infty} d^*(x_n, -1) = \lim_{n\to\infty} d^*|x_n - (-1)| = \lim_{n\to\infty} \left|\frac{1-n}{n} + 1\right| = 0$$

and $d^*(-1,-1) = -1$.

Lemma 2.7 (see [24]) Let $(\mathcal{U}, d^*)$ be a dualistic partial metric space.

(1) Every Cauchy sequence in $(\mathcal{U}, \rho^m_{d^*})$ is also a Cauchy sequence in $(\mathcal{U}, d^*)$.
(2) A dualistic partial metric $(\mathcal{U}, d^*)$ is complete if and only if the induced metric space $(\mathcal{U}, \rho^m_{d^*})$ is complete.
(3) A sequence $\{x_n\}$ in $\mathcal{U}$ converges to a point $x \in \mathcal{U}$ with respect to $\tau(\rho^m_{d^*})$ if and only if $d^*(x,x) = \lim_{n\to\infty} d^*(x_n, x) = \lim_{n\to\infty} d^*(x_n, x_m)$.

Then $\mathcal{P}$ is a parametric metric on $\mathcal{U}$ and the pair $(\mathcal{U}, \mathcal{P})$ is a parametric metric space. Let $\mathcal{U}$ be a set. A point $z \in \mathcal{U}$ is a point of coincidence of a pair of self-maps $f, g: \mathcal{U} \to \mathcal{U}$ and $\theta \in \mathcal{U}$ is its coincidence point if $f\theta = g\theta = z$. Mappings f and g are weakly compatible if $fg\theta = gf\theta$ for each of their coincidence points θ [25,26,27] and occasionally weakly compatible if the same holds for some coincidence point [28]. The set of fixed points of a self-map $f: \mathcal{U} \to \mathcal{U}$ will be denoted as $\mathfrak{F}(f)$. The mapping f is said to possess property (P) if $\mathfrak{F}(f^n) = \mathfrak{F}(f)$ for each $n \in \mathbb{N}$ (see [25,29]). A pair of self-maps $f, g: \mathcal{U} \to \mathcal{U}$ is said to have property (Q) if $\mathfrak{F}(f^n) \cap \mathfrak{F}(g^n) = \mathfrak{F}(f) \cap \mathfrak{F}(g)$ holds for each $n \in \mathbb{N}$ (see [25]).

Definition 2.8 (see [30]) Let $(\mathcal{U}, d^*)$ be a dualistic partial metric space and $f: \mathcal{U} \to \mathcal{U}$. Then f is called a dualistic expanding mapping, if for every $x, y \in \mathcal{U}$, there exists a number $\lambda > 1$ such that

$$|d^*(fx, fy)| \geq \lambda |d^*(x, y)|.$$

3 Main Results

Our main result as follows.

Theorem 3.1 Let $(\mathcal{U}, d^*)$ be a complete dualistic partial metric space and $f, g: \mathcal{U} \to \mathcal{U}$ be two maps such that $f(\mathcal{U}) \supset g(\mathcal{U})$ and that at least one of these subspaces is complete. Suppose that there exist real numbers α, β, γ satisfying $\beta, \gamma \geq 0$ and $\alpha > 1, \gamma < 1$ such that

$$|d^*(fx, fy)| \geq \alpha |d(gx, gy)| + \beta |d^*(gx, fx)| + \gamma |d^*(gy, fy)|, \tag{3.1}$$

$\forall x, y \in \mathcal{U}$. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is (occasionally) weakly compatible, then f and g have a unique common fixed point.

Proof Take arbitrary $x_0 \in \mathfrak{D}$. Construct sequences $\{x_n\}$ and $\{y_n\}$ such that $y_n = gx_n = fx_{n+1}$ for $n = 0, 1, 2, \ldots$. Applying (3.1), we obtain

$$\begin{aligned} |d^*(y_n, y_{n-1})| &= |d^*(fx_{n+1}, fx_n)| \\ &\geq \alpha |d(gx_{n+1}, gx_n)| + \beta |d^*(gx_{n+1}, fx_{n+1})| + \gamma |d^*(gx_n, fx_n)| \\ &= \alpha |d(y_{n+1}, y_n)| + \beta |d^*(y_{n+1}, y_n)| + \gamma |d^*(y_n, y_{n-1})| \\ &= (\alpha + \beta) |d^*(y_{n+1}, y_n)| + \gamma |d^*(y_n, y_{n-1})| \end{aligned}$$

Hence

$$(1 - \gamma)\, |d^*(y_{n-1}, y_n)| \geq (\alpha + \beta) |d^*(y_n, y_{n+1})| \tag{3.2}$$

Consequently,

$$|d^*(y_n, y_{n+1})| \leq \mu|d^*(y_{n-1}, y_n)| \tag{3.3}$$

where $\mu = \frac{1-\gamma}{\alpha+\beta} < 1$ for all $n \in \mathbb{N} \cup \{0\}$ and $t > 0$. Repeating (3.3) r-times, we get

$$|d^*(y_n, y_{n+1})| \leq \mu^n \, |d^*(y_0, y_1)|. \tag{3.4}$$

Now,

$$|d^*(y_n, y_n)| = |d^*(fx_{n+1}, fx_{n+1})|$$

$$\begin{aligned} &\geq \alpha|d(gx_{n+1}, gx_{n+1})| + \beta|d^*(gx_{n+1}, fx_{n+1})| + \gamma|d^*(gx_{n+1}, fx_{n+1})| \\ &= \alpha|d(y_{n+1}, y_{n+1})| + \beta|d^*(y_{n+1}, y_n)| + \gamma|d^*(y_{n+1}, y_n)| \\ &= (\beta+\gamma)|d^*(y_{n+1}, y_n)| + \alpha|d(y_{n+1}, y_{n+1})| \end{aligned}$$

The last inequality gives

$$|d(y_{n+1}, y_{n+1})| \leq \frac{1}{\alpha}|d^*(y_n, y_n)| - \frac{(\beta+\gamma)}{\alpha}|d^*(y_{n+1}, y_n)|. \tag{3.5}$$

Due to inequality (3.4), we have

$$|d(y_{n+1}, y_{n+1})| \leq \frac{1}{\alpha}|d^*(y_n, y_n)| - \frac{(\beta+\gamma)}{\alpha}\mu^n|d^*(y_0, y_1)|. \tag{3.6}$$

Similarly,

$$|d^*(y_n, y_n)| \leq \frac{1}{\alpha}|d^*(y_{n-1}, y_{n-1})| - \frac{(\beta+\gamma)}{\alpha}\mu^{n-1}|d^*(y_0, y_1)|. \tag{3.7}$$

The inequality (3.4) implies that

$$\begin{aligned} |d(y_{n+1}, y_{n+1})| &\leq \frac{1}{\alpha}\left\{\frac{1}{\alpha}|d^*(y_{n-1}, y_{n-1})| - \frac{(\beta+\gamma)}{\alpha}\mu^{n-1}|d^*(y_0, y_1)|\right\} \\ &-\frac{(\beta+\gamma)}{\alpha}\mu^n|d^*(y_0, y_1)| \\ &= \frac{1}{\alpha^2}|d^*(y_{n-1}, y_{n-1})| - \frac{(\beta+\gamma)}{\alpha^2}\mu^{n-1}|d^*(y_0, y_1)| \qquad - \frac{(\beta+\gamma)}{\alpha}\mu^n|d^*(y_0, y_1)| \\ &= \frac{1}{\alpha^2}|d^*(y_{n-1}, y_{n-1})| - (\beta+\gamma)\left[\frac{\mu^{n-1}}{\alpha^2} + \frac{\mu^n}{\alpha}\right]|d^*(y_0, y_1)| \\ &\leq \frac{1}{\alpha^3}|d^*(y_{n-2}, y_{n-2})| - (\beta+\gamma)\left[\frac{\mu^{n-2}}{\alpha^3} + \frac{\mu^{n-1}}{\alpha^2} + \frac{\mu^n}{\alpha}\right]|d^*(y_0, y_1)|. \end{aligned}$$

Continuing further, we get

$$\begin{aligned} |d(y_{n+1}, y_{n+1})| &\leq \frac{1}{\alpha^{n+1}}|d^*(y_0, y_0)| - (\beta+\gamma)\left[\frac{1}{\alpha^{n+1}} + \frac{\mu}{\alpha^n} + \cdots + \frac{\mu^n}{\alpha}\right]|d^*(y_0, y_1)| \\ &\leq \delta^{n+1}|d^*(y_0, y_0)| - (\beta+\gamma)[\delta^{n+1} + \delta^n\mu + \cdots + \delta\mu^n]|d^*(y_0, y_1)| \\ &= \delta^{n+1}|d^*(y_0, y_0)| - (\beta+\gamma)\rho^{n+1}|d^*(y_0, y_1)| \\ &\leq \delta^{n+1}|d^*(y_0, y_0)| + \rho^{n+1}|d^*(y_0, y_1)|, \end{aligned} \tag{3.8}$$

where $\delta = \frac{1}{\alpha}$ and $\rho^{n+1} = \delta^{n+1} + \delta^n\mu + \cdots + \delta\mu^n$.

We deduce from (2.2) that

$$\rho_{d^*}(y_n, y_{n+1}) \leq |d^*(y_n, y_{n+1})| - d^*(y_n, y_n) \tag{3.9}$$

$$\leq |d^*(y_n, y_{n+1})| + d^*(y_n, y_n)$$
$$\leq \mu^n |d^*(y_0, y_1)| + \delta^n |d^*(y_0, y_0)| + \rho^n |d^*(y_0, y_1)|$$
$$= (\mu^n + \rho^n)|d^*(x_0, x_1)| + \delta^n |d^*(x_0, x_0)|.$$

Now, for $m > n$, we have

$$\begin{aligned}
\rho_{d^*}(y_n, y_m) &\leq \rho_{d^*}(y_n, y_{n+1}) + \rho_{d^*}(y_{n+1}, y_{n+2}) + \cdots + \rho_{d^*}(y_{m-1}, y_m) \\
&\leq (\mu^n + \rho^n)|d^*(y_0, y_1)| + \delta^n |d^*(y_0, y_0)| \\
&+(\mu^{n+1} + \rho^{n+1})|d^*(y_0, y_1)| + \delta^{n+1}|d^*(y_0, y_0)| + \cdots \\
&+(\mu^{m-1} + \rho^{m-1})|d^*(y_0, y_1)| + \delta^{m-1}|d^*(y_0, y_0)| \\
&= (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1})|d^*(y_0, y_1)| \\
&+(\rho^n + \rho^{n+1} + \cdots + \rho^{m-1})|d^*(y_0, y_1)| \\
&+(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1})|d^*(y_0, y_0)| \\
&\leq (\mu^n + \mu^{n+1} + \cdots + \mu^{m-1} + \cdots)|d^*(y_0, y_1)| \\
&+(\rho^n + \rho^{n+1} + \cdots + \rho^{m-1} + \cdots)|d^*(y_0, y_1)| \\
&+(\delta^n + \delta^{n+1} + \cdots + \delta^{m-1} + \cdots)|d^*(y_0, y_0)| \\
&= \frac{\mu^n}{1-\mu}|d^*(y_0, y_1)| + \frac{\rho^n}{1-\rho}|d^*(y_0, y_1)| + \frac{\delta^n}{1-\delta}|d^*(y_0, y_0)|.
\end{aligned}$$

Hence

$$\rho_{d^*}(y_n, y_m) \leq \tfrac{\mu^n}{1-\mu}|d^*(y_0, y_1)| + \tfrac{\rho^n}{1-\rho}|d^*(y_0, y_1)| + \tfrac{\delta^n}{1-\delta}|d^*(y_0, y_0)|. \tag{3.10}$$

As $m, n \to \infty$, $\rho^m_{d^*}(y_n, y_m) = \max\{\rho_{d^*}(y_n, y_m), \rho_{d^*}(y_m, y_n)\} \to 0$, thus, $\{y_n\}$ is a Cauchy sequence in $(\mathcal{U}, \rho^m_{d^*})$. Since $(\mathcal{U}, d^*)$ is a complete dualistic partial metric space, by Lemma 2.1(2), $(\mathcal{U}, \rho^m_{d^*})$ is a complete metric space. Thus, there exists $z \in (\mathcal{U}, \rho^m_{d^*})$ such that $y_n = gx_n = fx_{n+1} \to z$ as $n \to \infty$, that is $\lim_{n\to\infty} \rho_{d^*}(y_n, z) = 0$ and by Lemma 2.1 (3), we know that

$$d^*(z, z) = \lim_{n\to\infty} d^*(y_n, z) = \lim_{n\to\infty} d^*(y_n, y_m). \tag{3.11}$$

Since, $\lim_{n\to\infty} \rho_{d^*}(y_n, z) = 0$, by (2.2) and (3.5), we have

$$d^*(z, z) = \lim_{n\to\infty} d^*(y_n, z) = \lim_{n\to\infty} d^*(y_n, y_m) = 0. \tag{3.12}$$

This shows that $\{y_n\}$ is a Cauchy sequence in $(\mathcal{U})$. Suppose that $f(\mathcal{U})$ is complete. Then, there exists $\theta \in \mathcal{U}$ such that $fx_n \to f\theta = z$. Let us prove that $f\theta = g\theta$. Then, using (3.1), we get

$$|d^*(y_n, f\theta)| = |d^*(fx_{n+1}, f\theta)|$$

$$\geq \alpha|d(gx_{n+1}, g\theta)| + \beta|d^*(gx_{n+1}, fx_{n+1})| + \gamma|d^*(g\theta, f\theta)|$$

which implies that as $n \to +\infty$,

$$0 \geq (\alpha + \gamma)|d(f\theta, g\theta)|$$

Hence $f\theta = g\theta$. Thus, $f\theta = g\theta = z$ is a point of coincidence for (f, g). Suppose that there is another point of coincidence $f\theta_1 = g\theta_1 = z_1$. Then

$$|d^*(z, z_1)| = |d^*(f\theta, f\theta_1)|$$

$$\geq \alpha|d(g\theta, g\theta_1)| + \beta|d^*(g\theta, f\theta)| + \gamma|d^*(g\theta_1, f\theta_1)|$$

implying (since $a > 1$) that $|d^*(z, z_1)| = 0 = |d^*(z, z)| = |d^*(z_1, z_1)|$. Consequently, $z = z_1$.Thus, the point of coincidence is unique. If the pair (f, g) is weakly compatible, applying [21, Proposition 1.12] we conclude that f and g have a unique common fixed point. If (f, g) is occasionally weakly compatible, the same conclusion follows from [22, Lemma 1.6]. This completes the proof.

Setting $\beta = 0 = \gamma$ in Theorem 3.1, we can obtain the following result.

Corollary 3.2 Let $(\mathcal{U}, d^*)$ be a complete dualistic partial metric space and $f, g: \mathcal{U} \to \mathcal{U}$ be two maps such that $f(\mathcal{U}) \supset g(\mathcal{U})$ and that at least one of these subspaces is complete. Suppose that there exists a real number $\alpha > 1$ such that

$$|d^*(fx, fy)| \geq \alpha |d(gx, gy)|, \tag{3.13}$$

$\forall x, y \in \mathcal{U}$. Then f and g have a unique point of coincidence. If, moreover, the pair (f, g) is (occasionally) weakly compatible, then f and g have a unique common fixed point.

Corollary 3.3 Let $(\mathcal{U}, d^*)$ be a complete dualistic partial metric space and $f: \mathcal{U} \to \mathcal{U}$ be a surjection. Suppose that there exists a real number $\alpha > 1$ such that

$$|d^*(fx, fy)| \geq \alpha |d(x, y)|, \tag{3.14}$$

$\forall x, y \in \mathcal{U}$. Then f has a unique fixed point.

Proof From Corollary 3.3, it follows that f has a fixed point z in $\mathcal{U}$ by setting $g = i_{\mathcal{U}}$.

Uniqueness. Suppose that $z \neq z^\star$ is also another fixed point of f, then from condition (3.14), we obtain

$$|d^*(z, z^\star)| = |d^*(fz, fz^\star)| \geq \alpha |d(z, z^\star)|$$

which implies $d^*(z, z^\star) = 0 = d^*(z, z) = d^*(z^\star, z^\star)$, that is $z = z^\star$. This completes the proof.

Corollary 3.4 Let $(\mathcal{U}, d^*)$ be a complete dualistic partial metric space and $f: \mathcal{U} \to \mathcal{U}$ be a surjection. Suppose that there exists a positive integer n and a real number a real number $\alpha > 1$ such that

$$|d^*(f^n x, f^n y)| \geq \alpha |d(x, y)|, \tag{3.15}$$

$\forall x, y \in \mathcal{U}$. Then f has a unique fixed point.

Proof From Corollary 3.3, f^n has a fixed pointz. But $f^n(fz) = f(f^n z) = fz$, So fz is also a fixed point of f^n. Hence $fz = z$, z is a fixed point of f. Since the fixed point of f is also fixed point of f^n, the fixed point of f is unique.

Corollary 3.5 (Corollary 2.1 of Huang et al. [2]) Let $(\mathcal{U}, d)$ be a complete partial metric space and $f: \mathcal{U} \to \mathcal{U}$ be a surjection. Suppose that there exists $\lambda > 1$ such that

$$d(fx, fy) \geq \lambda d(x, y), \tag{3.16}$$

$\forall x, y \in \mathcal{U}$, then f has a unique fixed point.

Proof Since the restriction of a dualistic partial metric d^* to $[0, \infty)$, $d^*\big|_{[0,\infty)} = d$ is a partial metric, so arguments follow the same lines as in the proof of Theorem 3.1.

Corollary 3.6 (Theorem 2.1 of Huang et al. [2]) Let $(\mathcal{U}, d)$ be a complete partial metric space and $f: \mathcal{U} \to \mathcal{U}$ be a surjection. Suppose that there exist real numbers α, β, γ satisfying $\beta, \gamma \geq 0$ and $\alpha > 1$ such that

$$d(fx, fy) \geq \alpha\, d(x, y) + \beta\, d(x, fx) + \gamma\, d(y, fy), \tag{3.17}$$

$\forall x, y \in \mathcal{U}$, then f has a unique fixed point.

Proof Set $d^*\big|_{[0,\infty)} = d$ and arguments follow the same lines as in the proof of Theorem 3.1.

Observations 3.7

1. Usually the range of a dualistic partial metric d^* is $(-\infty,\infty)$ but if we replace $(-\infty,\infty)$ by $[0,\infty)$, then d^* is identical to a partial metric d and hence Theorem 3.1 is applicable in the setting of partial metric space.
2. If we set $d(\mathrm{x},\mathrm{x}) = 0$ in Corollary 3.5 and Corollary 3.6, we retrieve corresponding theorems in metric spaces.
3. Our main result extends and generalizes some well-known results of Huang et al. [2] and Wang et al. [3].

Example 3.8 Define $d^*: (-\infty, 0] \times (-\infty, 0] \to (-\infty,\infty)$ by $d^*(\mathrm{x},\mathrm{y}) = \max\{\mathrm{x},\mathrm{y}\}$. It is easy to check that $\left((-\infty, 0], d^*\right)$ is a complete dualistic partial metric space. Define $\mathrm{f},\mathrm{g} : (-\infty, 0] \to (-\infty, 0]$ as $\mathrm{fx} = 6\mathrm{x}, \mathrm{gx} = 3\mathrm{x}, \forall\, \mathrm{x} \in (-\infty, 0]$. Further, for all $\mathrm{x},\mathrm{y} \in (-\infty, 0]$ with $\mathrm{x} \geq \mathrm{y}$, and $\alpha = 2$, we have

$$
\begin{aligned}
|d^*(\mathrm{fx},\mathrm{fy})| &= |\max\{6\mathrm{x}, 6\mathrm{y}\}| = |6\mathrm{x}| \\
&\geq 2|\max\{3\mathrm{x}, 3\mathrm{y}\}| \\
&= 2|d^*(\mathrm{x},\mathrm{y})|.
\end{aligned}
$$

Cleary, (3.13) is satisfied and f is a self-surjection on $(-\infty, 0]$. In the view of Corollary 3.1, f and g has a unique common fixed point in $(-\infty, 0]$, indeed $\mathrm{f}0 = \mathrm{g}0 = 0$. Also
$|d^*(\mathrm{fx},\mathrm{fy})| = |\max\{6\mathrm{x}, 6\mathrm{y}\}| = |6\mathrm{x}| > \alpha|\mathrm{x}| = \alpha|\max\{\mathrm{x},\mathrm{y}\}| = \alpha|d^*(\mathrm{x},\mathrm{y})|$,
for $1 < \alpha < 6$ and $\forall\, \mathrm{x},\mathrm{y} \in (-\infty, 0]$ with $\mathrm{x} \geq \mathrm{y}$. Thus, f is a dualistic expanding surjective self-mapping on $(-\infty, 0]$. From, Corollary 3.3, $\mathrm{f}0 = 0 \in (-\infty, 0]$ is unique.

Disclaimer (Artificial Intelligence)

Author(s) hereby declare that NO generative AI technologies such as Large Language Models (ChatGPT, COPILOT, etc.) and text-to-image generators have been used during writing or editing of manuscripts.

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Competing Interests

Authors have declared that no competing interests exist.

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