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Undecidability of the Criterion for Recognizing the Fibonacci Numbers

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Abstract

Assuming the conjecture that all effectively calculable functions must be Turing-machine computable, this paper demonstrates that the problem of recognizing Fibonacci numbers must be in general undecidable. This means that there is no algorithm, which in a finite number of steps can correctly decide whether a given positive integer z of arbitrarily large size belongs to the Fibonacci sequence.

Keywords: Fibonacci numbers; golden ratio; Church-Turing thesis; Turing-machine; undecidability.

1 Introduction

Suppose we are given a positive integer z of arbitrary size and asked whether this integer z is exactly a Fibonacci number $F(n)$.

Using Binet's formula [1,2], i.e., the closed-form expression for Fibonacci numbers,

$$F(n) = \frac{\varphi^n - (1-\varphi)^n}{\sqrt{5}} = \frac{\varphi^n - (1-\varphi)^n}{\varphi - (1-\varphi)} \quad , \tag{1}$$

where φ is the golden ratio, one can derive the criterion that z is a Fibonacci number if and only if the closed interval S

$$S = \left[\varphi z - \frac{1}{z}, \varphi z + \frac{1}{z}\right] \tag{2}$$

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Intersects the set of all natural numbers ℕ at some element (or elements), that is

$$S \cap \mathbb{N} \neq \emptyset \quad . \tag{3}$$

In this short paper we will show that the criterion (3) is generally undecidable (that is, noncomputable in the sense of Church-Turing thesis of computability [3]), meaning that there is no algorithm which can in a finite number of steps accurately decide whether or not an arbitrary positive integer z is a Fibonacci number.

2 The Golden Ratio

Since the golden ratio itself is defined by

$$\varphi^2 - \varphi - 1 = 0 \tag{4}$$

(Meaning that φ is an algebraic number of degree 2), its exact value is

$$\varphi = \frac{1}{2}\left(1 + \sqrt{5}\right) \quad , \tag{5}$$

and in accordance with [4] its decimal expansion is

$$\varphi = 1.6180339887498948482045868343656381177203091798057628 6\ldots \quad . \tag{6}$$

The golden ratio φ is the most irrational among irrational numbers because its continued fraction representation

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}} \tag{7}$$

converges more slowly than any other continued fraction [5].

Nevertheless, as much as all other algebraic numbers, the golden ratio φ is computable. To be sure, as φ is given by the infinite series (expanded via a Taylor series) [6].

$$\varphi = \frac{13}{8} + \sum_{m=0}^{\infty} \frac{(-1)^{(m+1)}(2m+1)!}{(m+2)!\, m!\, 4^{(2m+3)}} \quad , \tag{8}$$

The golden ratio φ can be found by way of the approximation.

$$\varphi \cong 1.609375 + \frac{1}{4^5}\left(1 - \frac{5}{2} \cdot \frac{1}{4^2} + \cdots + (-1)^{(k+1)} \frac{(2k+1)!}{(k+2)!\, k!} \cdot \frac{1}{4^{(2k-2)}}\right) \tag{9}$$

where for any $k \geq 1$ the remainder term R_k must be less than some desired value

$$|R_k| = \frac{(k+3)(k+4)\cdots(2k+1)}{k!} \cdot \frac{1}{4^{(2k+3)}} < 10^{-N} \ \ (N > 1) \tag{10}$$

In order to ensure that a decimal expression of φ would be correct up to N decimal places.

This implies that the golden ratio φ can be computed: Namely, according to the expression (9), there is an exact algorithm, which for any given number N provides the corresponding value of φ (despite the fact that in the limit $N \to \infty$ this algorithm would run forever).

3 The Criterion Recognizing the Fibonacci Numbers

But unlike the expression (9), the criterion (3) is generally non-computable.

To show this, let us first present the criterion (3) in the form of the equality

$$\left\lfloor \varphi z + \frac{1}{z} \right\rfloor - \left\lceil \varphi z - \frac{1}{z} \right\rceil = 0 \quad , \tag{11}$$

where $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ stand for the floor and ceiling functions respectively. Assume that the golden ratio $\varphi = 1 + \{\varphi\}$ (where $\{\varphi\}$ is its fractional part) has been calculated to the accuracy of N decimal places, that is,

$$\{\varphi\} = \sum_{i=1}^{N} a_i \cdot 10^{-i} \text{ where } a_i \in \{0,1,\dots,9\} \quad . \tag{12}$$

Substituting (12) into (11) one can find

$$\left\lfloor z \sum_{i=1}^{N} a_i \cdot 10^{-i} + z + \frac{1}{z} \right\rfloor - \left\lceil z \sum_{i=1}^{N} a_i \cdot 10^{-i} + z - \frac{1}{z} \right\rceil = 0 \quad . \tag{13}$$

Suppose that the given positive integer z make the equality (13) true. The question is does this mean that z is a Fibonacci number?

As a matter of fact, given the limited computational resources we have (especially, time) we can never be sure in the positive answer to this question because we can never know whether the equality (13) would still hold true if the decimal representation of φ took more decimal places, that is, if $\{\varphi\}$ were computed to a higher accuracy of $N' > N$.

For example, in the case that the computed value of the golden ratio has only three decimals $\{\varphi\} = 0.618$, the equality (13) would hold for the integer $z = 1000$ (and thus 1000 would be wrongly found to be a Fibonacci number), while with eight computed decimals a_i of φ (i.e., when $\{\varphi\} = 0.61803398$) the same $z = 1000$ would break down the equality (13).

The problem, of course, is that defining the property of being a Fibonacci number for an arbitrary positive integer z demands knowing either the whole Fibonacci sequence (to compare its members with z) or the exact value of the golden ratio φ (to check whether the equality (13) holds for z). Obviously, in both cases it would involve quantification over an infinite domain and thus undecidability.

Indeed, suppose that the procedure for finding the value of φ and afterwards checking the equality (13) for the given z runs on a classical Turing machine (TM) [7]. It is clear that if we want the exact answer and the positive integer z is unlimited in its size, then the accuracy of φ must be unlimited as well (i.e., in the decimal representation (12) the number $N = \infty$) and consequently this procedure would run forever. Also, because a quantum TM can only explore a finite domain in a finite time and, hence, is no better than a classical TM in terms of computability (though it may be more efficient [8]), this procedure running on a quantum TM would hold forever too.

Thus, assuming the conjecture that all effectively calculable functions must be TM-computable [9], we find that the problem of recognizing Fibonacci numbers must be in general undecidable. This means that there is no algorithm, which in a finite number of steps can correctly decide whether a given positive integer z of arbitrarily large size belongs to the Fibonacci sequence.

4 Conclusion Remarks

Providing a special role the Fibonacci sequence may play in nature, the gotten above conclusion seems troubling. Indeed, why do so many natural patterns reflect the Fibonacci sequence (for example, as reported by [10], phyllotaxic patterns that are generated each time when a vascular plant repeatedly produces similar botanical elements at its tip such as leaves, bractae, florets etc. are directly related to the Fibonacci sequence and the golden ratio φ) if the problem of recognizing the Fibonacci numbers is generally undecidable?

For this reason, let us make a list of possible strategies for overcoming this problem and provide a brief comment on each of them.

1. *Since the assumption of infinite sizes is never realized in nature, one can suppose that all the integers that are related to natural processes are limited in size. Therefore, naturally originated positive integers can be recognizable as Fibonacci numbers in a finite amount of time.*

Much as this contrivance might seem pragmatically sound, it actually contradicts to one of the basic requirements of the procedure for a computable function. Indeed, if values of physical quantities are the outcomes of the corresponding computable functions then the procedures for computing these functions must theoretically work for arbitrarily large arguments [11]. Accordingly, the proposed contrivance implies that there exists a fundamental limit on computability of physical quantities.

However, even if one puts forward such a demarcation between computable and non-computable naturally originated positive integers, it would be bound to be ad hoc; in other words, no matter where the limit of computability would be drawn, it would always be an arbitrary "limit" perpetually subject to shifting (unless, of course, that limit is caused by some unknown yet physical law).

2. *Natural events go beyond normal Turing computation and so the values of physical quantities are computable with a hypercomputer, i.e., an information-processing machine able to achieve more than the universal TM.*

For example, the model of hypercomputation as an infinite time TM can correctly recognize whether an arbitrary positive integer z belongs to the Fibonacci sequence in a finite amount of time. However, whether the universe really admits the existence of a system, whose computational power strictly exceeds that of a TM, is unknown (see, for instance, the works debating the physical possibility of hypercomputers [12,13,14, 15,16,17]).

What's more, the evidences such as the constant failure to solve the Halting problem physically and the success of simulating a plethora of physical systems on a TM point toward the negative answer to the question about the existence of hypercomputers.

3. *Much of what is presented about the role of the Fibonacci sequence and the golden ratio in various fields of science is false or seriously misleading (see, for example, the paper* [18] *that discusses some of the most repeated misconceptions about the golden ratio). Only an approximate equality between values of physical quantities and the Fibonacci numbers is possible.*

Even supposing that the concept of the predominance of the Fibonacci sequence in nature is misleading or at least exaggerated, the problem of the exact representation of the algebraic reals (as opposed to the computable reals) in different fields of science would remain. Along these lines, the question whether computable numbers can be used instead of the reals in a particular theory modeling natural events would be actual regardless of the status of the Fibonacci sequence in nature.

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Competing Interests

Author has declared that no competing interests exist.

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