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On Enumeration of Some Semigraphs

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

A semigraph G is vertex-labeled semigraph, if its n-vertices are labeled by distinct symbols. In this paper various results on enumeration of vertex-labeled semigraphs containing non-adjacent edges and number of vertex-labeled semigraphs with two adjacent s-edges are obtained. Also the number of vertex-labeled semigraphs on 1 to 8 vertices is calculated.

Keywords: Vertex-labeled semigraph; enumeration of vertex-labeled semigraphs.

1 Introduction

Semigraph is a discrete structure introduced by E. Sampatkumar [1] generalizing the graph [2]. Many concepts of graph theory are generalized in several variants of semigraphs. In the semigraph, every edge is an n-tupple for $n \geq 2$ and any two edges can have only one vertex in common. Therefore, many chemical structures resemble semigraph. Semigraphs have been well studied by the several authors like [3,4,5,6].

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One of the most important areas in graph theory is enumeration [7,8,9,10], which is the result of the pioneering work of Arthur Cayley. Graph enumeration is widely used in chemistry, physics, biology, information theory and so on [11,12]. In this paper various results on enumeration of the vertex-labeled semigraphs are established and also the total number of such semigraphs on 1 to 8 vertices is counted.

2 Basic Definitions

Following are the definitions related to the semigraphs. For more terminologies [1] may be referred.

Definition 2.1 [1]: A semigraph *G* is an ordered pair (V, X) where V is a non-empty set, whose elements are called *vertices* of *G* and the set *X* is the set of n –tuples, called the *edges* of *G*, of distinct vertices, for various $n \geq 2$, with the following conditions:

SG1: Any two edges have at most one vertex in common.

SG2: Two edges $(u_1, u_2, \ldots, u_n)$ and $(v_1, v_2, \ldots, v_m)$ are considered to be equal if and only if

i) $m = n$ and
ii) either $u_i = v_i$ or $u_i = v_{n-i+1}$ for $i = 1,2,3,\ldots n$.

Thus the edge $(u_1, u_2, \ldots, u_n)$ is the same as the edge $(u_n, u_{n-1}, \ldots, u_1)$.

Let *G* = (V, X) be a semigraph and $E = (v_1, v_2, \ldots, v_{n-1}, v_n)$ is an edge of *G*. Then the vertices v_1 and v_n are called the *end vertices* of *E*, represented by thick dots, the vertices $v_2, \ldots, v_{n-1}$ are called the *middle vertices or m–vertices* of *E*, represented by small hollow circles. A vertex v in *G* which appears as an end vertex of one edge and the middle vertex of the other edge is known as the *middle–cum–end vertex or ((m,e)) vertex,* represented by a small tangent to the hollow circle of middle vertex.

Example 2.2: Let *G* = (V, X) be a semigraph (Fig. 1), where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9, v_{10}\}$ and $X = \{(v_1, v_2, v_3), (v_3, v_4, v_5), (v_5, v_6, v_7, v_8), (v_8, v_3), (v_4, v_{10})\}$. In *G*, v_1, v_3, v_5, v_8, v_{10} are end vertices, v_2, v_6 and v_7 are middle vertices, v_4 is a middle–cum–end vertex and v_9 is an isolated vertex.

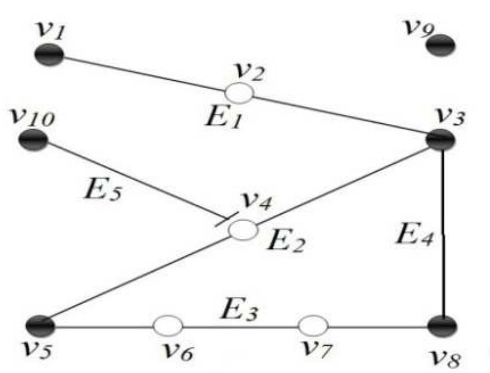

Fig. 1. Semigraph G

Definition 2.3 [1]: Two vertices in a semigraph *G* are said to be adjacent if they belong to the same edge and are consecutively adjacent if in addition they are consecutive in order as well. For example, in the Fig. 1, v_5, v_7 are adjacent vertices and v_6, v_7 are consecutive adjacent vertices.

Definition 2.4 [1]: Any two edges in a semigraph are said to be adjacent if they have a vertex in common. For example, in the Fig. 1, the edges E_1 and E_2 are adjacent and the edges E_1 and E_3 are non-adjacent.

Definition 2.5 [1]: Any two edges in a semigraph are said to be i) ee-adjacent if common vertex of the edges is end vertex in both the edges, ii) em-adjacent if common vertex of the edges is an end vertex of first edge and middle vertex of the second edge and iii) mm-adjacent if common vertex of the edges is a middle vertex in both the edges.

Definition 2.6 [1]: Cardinality of an edge in a semigraph is said to be k if the edge contains k number of vertices.

Definition 2.7 [1]: An edge in a semigraph *G* is said to be an s-edge if its cardinality $k \geq 3$.

Definition 2.8 [1]: A semigraph *G* is vertex-labeled semigraph, if its n-vertices are labeled by distinct symbols, such as $v_1, v_2, v_3, \ldots, v_n$.

3 Results and Discussion

Here we establish various results on enumeration of vertex-labeled semigraphs containing non-adjacent s-edges.

Note that $\binom{n}{r}$ value is considered as zero when $r > n$ in the following theorems.

Theorem 3.1: The number of vertex-labeled semigraphs on n-vertices (and n distinct labels) with exactly one s-edge of cardinality k is $\binom{n}{r}\frac{k!}{2}$, where $3 \leq k \leq n$.

Proof: Consider a semigraph on n-vertices with exactly one s-edge of cardinality $3 \leq k \leq n$ and the remaining vertices (if any) are isolated vertices.

Then the k vertices of the s-edge can be labeled by choosing k labels from n-labels in $\binom{n}{k}$ ways and the $(n-k)$ isolated vertices are labeled by exactly one way by using the remaining $(n-k)$ labels.

As an edge in a semigraph is symmetric, the permutation of these selected k-labels gives $\frac{k!}{2}$ different labeling of the s-edge.

Hence the number of vertex labeled semigraphs on n-vertices, with exactly one s-edge of cardinality $3 \leq k \leq n$ is $\binom{n}{k}\frac{k!}{2}$.

Theorem 3.2: The number of vertex-labeled semigraphs on n -vertices with exactly one s-edge of cardinality $k=n-1$ and the remaining edges of cardinality 2 is

$$=\binom{n}{n-1}\frac{(n-1)!}{2}2^{n-1}=\binom{n}{k}\frac{k!}{2}2^{k} \quad \text{where } k=n-1\geq 3.$$

Proof: Consider a semigraph *G* on n -vertices with exactly one s-edge of cardinality $k=n-1$ and the remaining edges (if any) of cardinality 2.

Then the k vertices of s-edge can be labeled by choosing k labels from n -labels in $\binom{n}{k}$ ways and the remaining 1 vertex is labeled by remaining label.

As an edge in a semigraph is symmetric, the permutation of these selected k -labels, gives $\frac{k!}{2}$ different labeling of the s-edge.

Clearly in this semigraph, we can add maximum $l=k(1)=k$ edges of cardinality 2 between k -vertices of s-edge and the remaining 1 vertex.

If the semigraph contains q edges $(q=0,1,2,3,\ldots l)$ of cardinality 2, then these q -edges can be adjacent to any q vertices of s-edge, which can be chosen in $\binom{k}{q}$ ways. Hence the number of semigraphs with q number of edges of cardinality 2 and one s-edge is $\binom{n}{k}\frac{k!}{2}\binom{k}{q}$.

Hence the total number of vertex-labeled semigraphs on n -vertices with exactly one s-edge of cardinality $k=n-1$ and the remaining edges of cardinality 2 are

$$=\binom{n}{k}\frac{k!}{2}\binom{k}{0}+\binom{n}{k}\frac{k!}{2}\binom{k}{1}+\binom{n}{k}\frac{k!}{2}\binom{k}{2}+\ldots+\binom{n}{k}\frac{k!}{2}\binom{k}{k}$$

$$=\binom{n}{k}\frac{k!}{2}\left\{\binom{k}{0}+\binom{k}{1}+\binom{k}{2}+\ldots+\binom{k}{k}\right\}$$

$$=\binom{n}{k}\frac{k!}{2}2^{k}.$$

Theorem 3.3: The number of vertex-labeled semigraphs on n -vertices with exactly one s-edge of cardinality k and the remaining edges of cardinality 2 (if any) is

$$=\binom{n}{k}\frac{k!}{2}2^{\binom{n-k}{2}+k(n-k)}, \text{ where } 3\leq k<n-1.$$

Proof: Consider a semigraph *G* on n -vertices with exactly one s-edge of cardinality $3 \le k < n-1$ and the remaining edges (if any) of cardinality 2.

Clearly such semigraph can have maximum $k(n-k)$ edges of cardinality 2 between k -vertices of s-edge and the remaining $(n-k)$ vertices and $\binom{n-k}{2}$ edges of cardinality 2 among $(n-k)$ vertices. Hence the maximum number of edges of cardinality 2 is

$$l = k(n-k) + \binom{n-k}{2}.$$

Hence the semigraph *G* can have q number of edges of cardinality 2 in $\binom{l}{q}$ ways.

Now the k vertices of s-edge can be labeled by choosing k labels from n -labels in $\binom{n}{k}$ ways and the remaining $(n-k)$ vertices are labeled by remaining labels.

As an edge in a semigraph is symmetric, the permutation of these selected k labels gives $\frac{k!}{2}$ different labeling of s-edge and hence different s-edges.

Therefore the number of vertex-labeled semigraphs on n -vertices with exactly one s-edge of cardinality $3 \le k < n-1$ and the q edges of cardinality 2 is $\binom{n}{k}\frac{k!}{2}\binom{l}{q}$.

Hence the total number of vertex-labeled semigraph on n -vertices with one s-edge of cardinality $3 \le k < n-1$ and the remaining $q = 0,1,2,3,\ldots l$ edges of cardinality 2 are

$$= \binom{n}{k}\frac{k!}{2}\binom{l}{0} + \binom{n}{k}\frac{k!}{2}\binom{l}{1} + \binom{n}{k}\frac{k!}{2}\binom{l}{2} + \ldots. + \binom{n}{k}\frac{k!}{2}\binom{l}{l}$$

$$= \binom{n}{k}\frac{k!}{2}\left\{\binom{l}{0} + \binom{l}{1} + \binom{l}{2} + \ldots. + \binom{l}{l}\right\}$$

$$= \binom{n}{k}\frac{k!}{2}2^{l}$$

$$= \binom{n}{k}\frac{k!}{2}2^{k(n-k)+\binom{n-k}{2}} .$$

Theorem 3.4: The number of vertex-labeled semigraphs on n-vertices with two non-adjacent s-edges of cardinalities $k_1 \geq 3, k_2 \geq 3$ with $6 \leq k_1 + k_2 \leq n$ and the remaining edges of cardinality 2 (if any) is

$$= \binom{n}{k_1} \frac{k_1!}{2} \binom{n-k_1}{k_2} \frac{k_2!}{2} 2^{\binom{n-k_1-k_2}{2} + (n-k_1-k_2)(k_1+k_2) + k_1 k_2}.$$

Proof: Consider a semigraph G on n-vertices with two non-adjacent s-edges E_1, E_2 of cardinalities $k_1 \geq 3, k_2 \geq 3$, $6 \leq k_1 + k_2 \leq n$ and the remaining edges (if any) are of cardinality 2.

Clearly such semigraph can have i) $(n-k_1-k_2)(k_1+k_2)$ edges of cardinality 2 between $k_1 + k_2$ vertices of two s-edges and the remaining $(n-k_1-k_2)$ vertices of the semigraph, ii) $\binom{n-k_1-k_2}{2}$ edges of cardinality 2 among the $(n-k_1-k_2)$ vertices and iii) $k_1 k_2$ edges between the s-edges *E_1, E_2*. Hence the total number of edges of cardinality 2 is

$$l = \binom{n-k_1-k_2}{2} + (n-k_1-k_2)(k_1+k_2) + k_1 k_2.$$

Hence the semigraph can have any q number of edges cardinality 2 in $\binom{l}{q}$ ways.

Now the k_1 vertices of s-edge E_1 can be labeled by choosing k_1 label from n-label in $\binom{n}{k_1}$ ways, k_2 vertices of s-edge E_2 can be labeled by choosing k_2 label from $(n-k_1)$ label in $\binom{n-k_1}{k_2}$ ways and remaining $(n-k_1-k_2)$ vertices are labeled by remaining labels.

As an edge in a semigraph is symmetric, the permutation of these selected k_1 and k_2 labels gives $\frac{k_1!}{2}$ and $\frac{k_2!}{2}$ different labeling of s-edges E_1, E_2 respectively and hence different labeling.

Therefore the number of vertex-labeled semigraphs on n-vertices with two s-edges of cardinalities k_1, k_2 and the q edges of cardinality 2 is $\binom{n}{k_1} \frac{k_1!}{2} \binom{n-k_1}{k_2} \frac{k_2!}{2} \binom{l}{q}$.

Hence the total number of labeled semigraph on n-vertices with two s-edges of cardinalities k_1, k_2 and the remaining $q = 0,1,2,3,\ldots l$ edges of cardinality 2 are

$$=\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\binom{l}{0}+\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\binom{l}{1}+\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\binom{l}{2}+\ldots+$$

$$+\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\binom{l}{l}$$

$$=\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\left\{\binom{l}{0}+\binom{l}{1}+\binom{l}{2}+\ldots+\binom{l}{l}\right\}$$

$$=\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}2^{l}$$

$$=\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}2^{\binom{n-k_1-k_2}{2}+(n-k_1-k_2)(k_1+k_2)+k_1k_2}.$$

Theorem 3.5: Let G be a semigraph on n -vertices with non-adjacent s-edges of cardinalities $k_1,k_2,k_3,\ldots k_i$ where $3\le k_1,k_2,k_3,\ldots,k_i\le n$ and $k_1+k_2+k_3+\ldots+k_i=t\le n$ and the remaining edges of cardinality 2 (if any), then the number of such labeled semigraphs is

$$=\binom{n}{k_1}\frac{k_1!}{2}\binom{n-k_1}{k_2}\frac{k_2!}{2}\binom{n-k_1-k_2}{k_3}\frac{k_3!}{2}\ldots\ldots\binom{n-k_1-k_2\ldots-k_{i-1}}{k_i}\frac{k_i!}{2}2^{\binom{n-t}{2}+t(n-t)+\sum_{l=1}^{i-1}\sum_{m=l+1}^{i}k_lk_m}.$$

Proof: Proof is similar to Theorem 3.4.

Now we state the theorem on enumeration of semigraphs containing two adjacent s-edges.

Theorem 3.6: The number of non isomorphic semigraphs on n -vertices with exactly two adjacent s-edges of cardinalities $k_1\ge 3$ and $k_2\ge 3$ with $5\le k_1+k_2-1\le n$ is

$$=\left\lceil\frac{k_1}{2}\right\rceil\left\lceil\frac{k_2}{2}\right\rceil \qquad \text{if } k_1\ne k_2$$

$$=\frac{1}{2}\left\lceil\frac{k}{2}\right\rceil\left\{\left\lceil\frac{k}{2}\right\rceil+1\right\} \quad \text{if } k_1=k_2=k.$$

Proof: Let G be a semigraph on n -vertices with two adjacent s-edges $E_1=(u_1,u_2,\ldots,u_{k_1-1},u_{k_1})$ and $E_2=(v_1,v_2,\ldots,v_{k_2-1},v_{k_2})$ of cardinalities k_1 and k_2 with $5\le k_1+k_2-1\le n$, $k_1,k_2\ge 3$.

First, we note that, in a semigraph, the edge $(u_1, u_2, \ldots, u_k)$ of cardinality k is same as the edge $(u_k, u_{k-1}, \ldots, u_1)$. Hence the vertices u_1 and u_{k_1} , u_2 and u_{k_1-1} , . . . are in the symmetric positions of the edge and the number of such symmetric positions is $\left\lfloor \frac{k}{2} \right\rfloor$. Therefore the number of non symmetric positions in an edge of cardinality k is $k - \left\lfloor \frac{k}{2} \right\rfloor = \left\lceil \frac{k}{2} \right\rceil$.

Now we define matrix $V = [a_{ij}]$, where the rows correspond to vertices u_i, $i = 1,2,\ldots k_1$ of the edge E_1 (in order) and the columns correspond to vertices v_j, $j = 1,2,\ldots k_2$ of the edge E_2 (in order) and hence each a_{ij} corresponds to the semigraph *G*, with common vertex of its two edges as u_i of E_1 and v_j of E_2. Clearly this $k_1 \times k_2$ matrix gives the all possible semigraphs with two adjacent edges.

Keeping in mind each a_{ij} of the matrix *V*, represents one semigraph and symmetric concepts discussed in the beginning of the proof, using SG2 of Definition 2.1, the semigraphs corresponding to last $\left\lfloor \frac{k_1}{2} \right\rfloor$ rows of the matrix *V* are isomorphic to the semigraphs corresponding to first $\left\lfloor \frac{k_1}{2} \right\rfloor$ rows of the matrix *V*. Similarly, the semigraphs corresponding to last $\left\lfloor \frac{k_2}{2} \right\rfloor$ columns of the matrix *V* are isomorphic to the semigraphs corresponding to first $\left\lfloor \frac{k_2}{2} \right\rfloor$ columns of the matrix *V*.

Let V' be the matrix obtained by deleting the last $\left\lfloor \frac{k_1}{2} \right\rfloor$ rows and the last $\left\lfloor \frac{k_2}{2} \right\rfloor$ columns of *V*.

Case i) If $k_1 \neq k_2$ then $a_{ij} \neq a_{ji}$ for all $i, j (i \neq j)$, that is all the elements of the matrix V' mutually non isomorphic semigraphs and hence, the number of non isomorphic semigraphs *G* is equal to the number of elements of the matrix V' .

Hence number of Semigraphs G with two adjacent edges of cardinalities k_1 and k_2 ($k_1 \neq k_2$) is

$$\left\{k_1 - \left\lfloor \frac{k_1}{2} \right\rfloor\right\}\left\{k_2 - \left\lfloor \frac{k_2}{2} \right\rfloor\right\} = \left\lceil \frac{k_1}{2} \right\rceil \left\lceil \frac{k_2}{2} \right\rceil.$$

Case ii) If $k_1 = k_2 = k$ then $a_{ij} = a_{ji}$ for all $i, j (i \neq j)$, that is the semigraph corresponding to a_{ij} is isomorphic to semigraph corresponding to a_{ji}. Hence the number of non isomorphic semigraphs *G* with two adjacent edges of cardinalities k_1 and k_2 with $k_1 = k_2 = k$ is

= the number of diagonal elements of V' + half the number of non diagonal elements of V'.

$$=\left\lceil \frac{k}{2} \right\rceil + \frac{1}{2}\left\{\left\lceil \frac{k}{2} \right\rceil\left\lceil \frac{k}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil\right\}$$

$$=\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\left\{\left\lceil \frac{k}{2} \right\rceil + 1\right\}.$$

We note the following and proceed to prove the Theorem 3.7.

Note: 1) If $k_1 \neq k_2$ then among the $\left\lceil \frac{k_1}{2} \right\rceil\left\lceil \frac{k_2}{2} \right\rceil$ non isomorphic semigraphs,

i) '1' semigraph is ee-adjacent,

ii) $\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}$ number of semigraphs are em-adjacent,

iii) $\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}$ number of semigraphs are me-adjacent and

iv) $\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}$ number of semigraphs are mm-adjacent.

2) If $k_1 = k_2 = k$ then among the $\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\left\{\left\lceil \frac{k}{2} \right\rceil + 1\right\}$ non isomorphic semigraphs,

i) '1' semigraph is ee-adjacent,

ii) $\left\{\left\lceil \frac{k}{2} \right\rceil - 1\right\}$ number of semigraphs are em-adjacent (isomorphic to me-adjacent),

iii) $\frac{1}{2}\left\lceil \frac{k}{2} \right\rceil\left\{\left\lceil \frac{k}{2} \right\rceil - 1\right\}$ number of semigraphs are mm-adjacent.

Theorem 3.7: The number of labeled semigraphs on n-vertices with exactly two adjacent s-edges of cardinalities $k_1 \geq 3$, $k_2 \geq 3$, $k_1 \neq k_2$ with $6 \leq k_1 + k_2 - 1 \leq n$ and the remaining edges of cardinality 2 (if any) is

$$=\binom{n}{k_1}\binom{n-k_1}{k_2-1}k_1!(k_2-1)!\left[1+\frac{1}{2}\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}+\frac{1}{2}\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}+\frac{1}{4}\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}\right]2^l$$

where

$$l = \binom{n-k_1-k_2+1}{2} + (n-k_1-k_2+1)(k_1+k_2-1) + (k_1-1)(k_2-1).$$

Proof: Let *G* be a semigraph on n -vertices with two adjacent s-edges $E_1 = (u_1, u_2, \ldots, u_{k_1 - 1}, u_{k_1})$ and $E_2 = (v_1, v_2, \ldots, v_{k_2 - 1}, v_{k_2})$ of cardinalities $k_1 \neq k_2$ with $6 \leq k_1 + k_2 - 1 \leq n$, $k_1, k_2 \geq 3$.

Then from the Theorem 3.6 and the above note, the number of non isomorphic semigraphs with two adjacent edges of cardinalities k_1 and k_2 is

Case i) '1' if E_1 and E_2 are ee-adjacent ,

Case ii) $\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}$ if E_1 and E_2 are em-adjacent,

Case iii) $\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}$ if E_1 and E_2 are me-adjacent and

Case iv) $\left\{\left\lceil \frac{k_1}{2} \right\rceil - 1\right\}\left\{\left\lceil \frac{k_2}{2} \right\rceil - 1\right\}$ if E_1 and E_2 are mm-adjacent

Clearly each of these four cases, semigraphs can have, i) $(n - k_1 - k_2 + 1)(k_1 + k_2 - 1)$ edges of cardinality 2 between $k_1 + k_2 - 1$ vertices of two s-edges and the remaining $(n - k_1 - k_2 - 1)$ vertices of the semigraph, ii) $\binom{n - k_1 - k_2 + 1}{2}$ edges of cardinality 2 among the $(n - k_1 - k_2 + 1)$ vertices and iii) $(k_1 - 1)(k_2 - 1)$ edges between the s-edges E_1, E_2 . Hence the total number of edges of cardinality 2 is

$$l = \binom{n - k_1 - k_2 + 1}{2} + (n - k_1 - k_2 + 1)(k_1 + k_2 - 1) + (k_1 - 1)(k_2 - 1).$$

Hence each of these semigraphs can have any *q* number of edges of cardinality 2 in $\binom{l}{q}$ ways.

Now k_1 vertices of the edge E_1 are labeled by choosing k_1 labels from n -labels in $\binom{n}{k_1}$ ways, the $k_2 - 1$ vertices of the edge E_2 are labeled by choosing k_2 labels from $n - k_1$ labels in $\binom{n - k_1}{k_2 - 1}$ ways and the remaining $(n - k_1 - k_2 + 1)$ vertices are labeled by exactly one way, using the remaining labels.

The permutation of these selected labels gives,

$k_1!(k_2 - 1)!$ different labeling of s-edges E_1 and E_2 for the case (i),

$\frac{1}{2} k_1!(k_2 - 1)!$ different labeling of s-edges E_1 and E_2 for the case (ii),

$\frac{1}{2}k_1!(k_2-1)!$ different labeling of s-edges E_1 and E_2 for the case (iii) and

$\frac{1}{4}k_1!(k_2-1)!$ different labeling of s-edges E_1 and E_2 for the case (iv).

Therefore the number of vertex-labeled semigraphs on n-vertices with two s-edges of cardinalities k_1, k_2 and the q edges of cardinality 2 is

$$=\binom{n}{k_1}\binom{n-k_1}{k_2-1}k_1!(k_2-1)!\binom{l}{q}+\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}\binom{n}{k_1}\binom{n-k_1}{k_2-1}\frac{(k_1)!}{2}(k_2-1)!\binom{l}{q}+$$

$$\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}\binom{n}{k_1}\binom{n-k_1}{k_2-1}\frac{(k_1)!}{2}(k_2-1)!\binom{l}{q}+$$

$$\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}\binom{n}{k_1}\binom{n-k_1}{k_2-1}\frac{(k_1)!}{2}\frac{(k_2-1)!}{2}\binom{l}{q}$$

$$=\binom{n}{k_1}\binom{n-k_1}{k_2-1}k_1!(k_2-1)!\left[1+\frac{1}{2}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}+\frac{1}{2}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}+\frac{1}{4}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}\right]\binom{l}{q}$$

Hence the total number of labeled semigraph on n-vertices with two adjacent s-edges of cardinalities k_1, k_2 $(k_1 \neq k_2)$ and the remaining $q = 0,1,2,3,\ldots l$ edges of cardinality 2 are

$$=\binom{n}{k_1}\binom{n-k_1}{k_2-1}k_1!(k_2-1)!\left[1+\frac{1}{2}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}+\frac{1}{2}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}+\frac{1}{4}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}\right]$$
$$\times\left\{\binom{l}{0}+\binom{l}{1}+\ldots+\binom{l}{l}\right\}$$

$$=\binom{n}{k_1}\binom{n-k_1}{k_2-1}k_1!(k_2-1)!\left[1+\frac{1}{2}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}+\frac{1}{2}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}+\frac{1}{4}\left\{\left\lceil\frac{k_1}{2}\right\rceil-1\right\}\left\{\left\lceil\frac{k_2}{2}\right\rceil-1\right\}\right]2^l$$

where

$$l=\binom{n-k_1-k_2+1}{2}+(n-k_1-k_2+1)(k_1+k_2-1)+(k_1-1)(k_2-1).$$

Theorem 3.8: The number of labeled semigraphs on n-vertices with exactly two adjacent s-edges of cardinalities $k_1 \geq 3$, $k_2 \geq 3$, $k_1 = k_2 = k$ with $5 \leq k_1 + k_2 - 1 \leq n$ and the remaining edges of cardinality 2 (if any) is

$$=\binom{n}{k}\binom{n-k}{k-1}k\,!(k\;-1)!\left[\frac{1}{2}+\frac{1}{2}\left\{\left\lceil\frac{k}{2}\right\rceil-1\right\}+\frac{1}{16}\left\lceil\frac{k}{2}\right\rceil\left\{\left\lceil\frac{k}{2}\right\rceil-1\right\}\right]2^l$$

Where

$$l = \binom{n-2k+1}{2} + (n-2k+1)(2k-1) + (k-1)^2 \cdot$$

Proof: Proof is similar to Theorem 3.7.

By using above results, the number vertex-labeled semigraphs containing at least one s-edge, on 3 to 8 vertices is tabulated in the Table 1.

Table 1. Number of vertex-labeled semigraph on 3 to 8 vertices

Number of vertices	Number of s-edges in semigraph	Cardinality (k) of s-edge / edges	Number of semigraph	Total number of semigraph on n-vertices
3	1	3	3	3
4	1	4	12	108
		3	96	
5	1	5	60	7020
		4	960	
		3	3840	
	2-adjacent edges	3, 3	2160	
6	1	6	360	960360
		5	11520	
		4	92160	
		3	245760	
	2-non adjacent edges	3, 3	92160	
	2-adjacent edges	3, 3	414720	
		4, 3	103680	
7	1	7	2520	232729560
		6	161280	
		5	2580480	
		4	13762560	
		3	27525120	
	2-non adjacent edges	3, 3	41287680	
		4, 3	5160960	
	2-adjacent edges	3,3	92897280	
		4, 3	46448640	
		4, 4	2903040	
8	1	8	20160	112354635456
		7	2580480	
		6	82575360	
		5	880803840	
		4	3523215360	
		3	5637144576	
	2-non adjacent edges	3,3	21139292160	
		4, 3	5284823040	
		4, 4	660602880	
		5, 3	330301440	
	2-adjacent edges	3, 3	31708938240	
		4, 3	23781703680	
		4, 4	2972712960	
		5, 3	3963617280	
		5, 4	495452160	
		6, 3	11890851840	

4 Conclusion

a) In the section 2, we defined basic terminologies needed to read this paper.
b) In the Theorem 3.1 to Theorem 3.3, we established results on enumeration of vertex-labeled semigraphs containing one s-edge and the remaining edges of cardinality 2 (if any)
c) In the Theorem 3.4, we established results on enumeration of vertex-labeled semigraphs containing two non-adjacent s-edge and the remaining edges of cardinality 2 (if any) and Theorem 3.5 generalizes the Theorem 3.4.
d) In the Theorem 3.6 to Theorem 3.8, we established results on enumeration of vertex-labeled semigraphs containing two adjacent s-edge and the remaining edges of cardinality 2 (if any).
e) Table 1 gives the number of labeled semigraphs on 3 to 8 vertices.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Sampathkumar E. Semigraphs and their application. Technical Report [DST/MS/022/94] Department of Science & Technology, Govt. of India. August; 1999.

[2] Frank Harary. Graph theory. Narosa Publishing House, New Delhi; 1988.

[3] Prabhakar R. Hampiholi, Jotiba P. Kitturkar. Strong circuit matrix and strong path matrix Semigraph. Annals of Pure and Applied Mathematics. 2015;10(2):247-254. (In press)

[4] Kamath SS, Hebbar SR. Strong and weak domination, full sets and domination balance in semigraph. Electronic Notes in Discrete Mathematics. 2003;15:112.

[5] Deshpande CM. Yogeshri Gaidhani, Athawale BP. Incidence matrix of a semigraph. British Journal of Mathematics & Computer Science. 2015;9(1):12-20.

[6] Sundareswaran R, Swaminathan V. (m, e)-domination in semigraphs. Electronic Notes in Discrete Mathematics. 2009;33:75–80.

[7] Narsingh D. Graph theory with applications to engineering and computer science. New Delhi: Prentice Hall of India Private Limited; 2000.

[8] Gallian JA. A dynamic survey of graph labeling. The Electronics Journal of Combinatorics. 2014;16.

[9] Modha MV, Kanani KK. On k-cordial labeling of some graphs. British Journal of Mathematics & Computer Science. 2016;13(3):1-7.

[10] Anil Kumar V, Vandana PT. V4 magic labelings of some shell related graphs. British Journal of Mathematics and Computer Science. 2015;9(3):199-223.

[11] Balaban AT, Harary F. Chemical graphs—V enumeration and proposed nomenclature of benzenoid cata-condensed polycyclic aromatic hydrocarbons Tetrahedron. 1968;24(6):2505–2516.

[12] Prasanna NL. Applications of graph labeling in communication networks. Orient. J. Comp. Sci. and Technol. 2014;7:1.

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