British Journal of Mathematics & Computer Science
14(2): 1-7, 2016, Article no.BJMCS.23013
ISSN: 2231-0851

SCIENCEDOMAIN *international*
www.sciencedomain.org

A Monopoly Bertrand Game under Uncertainty

M. F. Elettreby[1,2*]

[1] *Department of Mathematics, Faculty of Science, King Khaled University, Abha 9004, Saudi Arabia.*
[2] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt.*

Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

Article Information

DOI: 10.9734/BJMCS/2016/23013
Editor(s):
(1) Jacek Dziok, Institute of Mathematics, University of Rzeszow, Poland.
Reviewers:
(1) Pu-Yan Nie, Guangdong University of Finance and Economics, China.
(2) A. A. Elsadany, Suez Canal University, Egypt.
(3) Na Wang, Hubei University, China.
(4) K. Gokulnath, Anna University, Chennai, India.
(5) Shi Kuiran, Nanjing University of Technology, China.
(6) Anonymous, Jaume I University, Spain.
Complete Peer review History: http://sciencedomain.org/review-history/13094

Received: ***10th November 2015***
Accepted: ***9th January 2016***
Published: ***27th January 2016***

Original Research Article

Abstract

In this paper, a bounded rational monopolist Bertrand model is proposed using a specific demand function. Therefore, some stability analysis is carried out to describe some complex dynamic phenomena such as bifurcation and chaos. Also, an approach for risk in the model which based on multi-objective method is studied.

Keywords: Bertrand; uncertainty; risk; equilibrium; asymptotic stability.

2010 Mathematics Subject Classification: 91Bxx.

**Corresponding author: E-mail: mohfathy@mans.edu.eg;*

1 Introduction

In economic structure, an economic market may be described as an oligopoly model where some firms produce the same goods, or homogeneous goods [1, 2]. This market is characterised by the interdependence between the firms. Each firm must take into account the actions of their rivals in the market. So that it can choose the best reaction [3, 4, 5, 6]. Game theory is one of the important tools for analyzing the strategic behaviour of this market [7, 8, 9, 10]. Various empirical works have shown that difference equations have been extensively used to simulate the behaviour of monopolistic markets [11, 12].

In economic markets, competed firms often face considerable uncertainty upon their productions. This uncertainty may concern with information on the exact demand function, rivals' costs or even components of firms own production cost. The impacts of uncertainty in the optimizing behavior of economic models have been previously examined by many researchers. Sandmo [13] has introduced a systematic study of the theory of firms under a price uncertainty and risk aversion. In [14], some properties of a class of models involving optimization under uncertainty have been analyzed. Dardononi [15] has presented the impact of uncertainty on an agent that has a two-argument utility function under plausible normality conditions. A simple non-cooperative games with common uncertainty in which the firms have two arguments utility functions has been discussed by Gradstein et al. [16], they shown that the effect of increased uncertainty on the equilibrium strategies is similar to its effect on the optimal strategy of a single agent. In literature, several studies concerning duopolistic models under uncertainty have investigated that in such models each firm is in risk neutral and may share or exchange its information on market uncertainty with its rival [17, 18, 19]. In these studies, it has been investigated how market uncertainty with either unknown market demand or unknown constant marginal cost can affect firms' behavior.

Taking a decision is associated with a risk due to its consequences. These consequences may be bad or undesirable. The amount of risk of the decision is related to the degree of uncertainty in the future circumstances. The cost of a certain good plays un important role in determining its price. Uncertainty measures how much the cost deviates from being constant. So, some papers consider the cost as a random variable instead of being constant [18]. Ahmed et al. [20], has introduced in detail an alternative approach for risk measurement. Since the main aim of any firm is to maximize its profit and at the same time to minimize the risk corresponding, he has used the standard deviation for modeling risk in Cournot game.

This paper introduces cost uncertainty into a simple Bertrand monopoly model. It assumed that the cost function used in this model is generated based on a random variable with zero mean and standard deviation equal to one. Therefore, with the given situation of uncertainty, the existence and uniqueness of the equilibrium point and its asymptotic behavior investigated. Furthermore, a Bertrand monopoly model under risk aversion is introduced.

The model is presented in section 2. The steady state and its local stability are studied in section 3. In section 4, a model with risk in multi-objective under uncertainty is presented.

2 The Model

In this section, a demand function is proposed [21, 22] in the form;

$$q_t = a - b p_t^2, \tag{2.1}$$

where $p_t > 0$ and $q_t > 0$ are the price and the quantity produced of a certain commodity at steps $t = 0, 1, 2, ...$ The positive constants a and b are the parameters of the model. The positivity of the quantity q_t requires that $p_t < \sqrt{a/b}$. Since a is the highest price in the market, then $a > c$ and $p_t > c$ for each quantity q_t, so, this means that $p_t \in (0, \sqrt{a/b})$.

A few approaches of uncertainty in multi-objective optimization problems have been reported in [18]. The most fundamental idea to address uncertainty in such problems is to begin by condensing random variables. Here, we used the following linear random cost function;

$$C(q_t) = (c + \gamma\,\epsilon)\,q_t, \tag{2.2}$$

where $c\,(> 0)$ is the fixed cost in the market. γ is a random parameter with zero mean and variance equal to 1. γ and $\epsilon\,(> 0)$ are the cost control parameters.

With this information and the information that the firm wants to maximize its profit, the profit will takes the form;

$$\begin{aligned}\Pi(p_t) &= p_t\,q_t - C(q_t) = (p_t - (c + \gamma\,\epsilon))\,q_t \\ &= (p_t - (c + \gamma\,\epsilon))\,(a - b\,p_t^2),\end{aligned} \tag{2.3}$$

which has the the following maximum value;

$$\Pi_{max} = \frac{2}{27}\left(-9\,a\,(c+\gamma\,\epsilon) + b\,(c+\gamma\,\epsilon)^3 + (3\,a + b\,(c+\gamma\,\epsilon)^2)\sqrt{(c+\gamma\,\epsilon)^2 + \frac{3\,a}{b}}\right),$$

at the price value $\overline{p} = \frac{b\,(c+\gamma\,\epsilon) + \sqrt{b^2\,(c+\gamma\,\epsilon)^2 + 3\,a\,b}}{3\,b}$.

A standard approach to generalize the static games to dynamic ones is called the bounded rationality approach [23]. The corresponding dynamical system for the model is given in the form;

$$p_{t+1} = p_t + \alpha(p_t)\,\frac{d\,\Pi(p_t)}{d\,p_t}, \tag{2.4}$$

where $\alpha(p_t) = \alpha$, is the speed of adjustment. Substituting Eq. (2.3) in Eq. (2.4), we get the following dynamical system;

$$p_{t+1} = p_t + \alpha\left(a + 2\,b\,(c+\gamma\,\epsilon)\,p_t - 3\,b\,p_t^2\right), \tag{2.5}$$

which is an one-dimensional nonlinear difference equation.

3 The Dynamical Analysis of the Model

The steady state solutions are important [24]. In the context of difference equations, a steady state solution $\overline{p}$ is defined to be the value that satisfies the relation $p_{t+1} = p_t = \overline{p}$ for all t. Then, we can get the steady state solutions for our model by the following. Substituting by $p_{t+1} = p_t = \overline{p}$ in Eq. (2.5), we get;

$$\overline{p} = \overline{p} + \alpha\left(a + 2\,b\,(c+\gamma\,\epsilon)\,\overline{p} - 3\,b\,\overline{p}^{\,2}\right). \tag{3.1}$$

From Eq. (3.1), there is a unique steady state $\overline{p} = \frac{b\,(c+\gamma\,\epsilon) + \sqrt{b^2\,(c+\gamma\,\epsilon)^2 + 3\,a\,b}}{3\,b}$, which is the same value that maximizes the profit. Local stability analysis [11] shows that the steady state $\overline{p}$ of the system $p_{t+1} = F(p_t)$ is stable iff;

$$\left|\frac{d\,F(p_t)}{d\,p_t}\right|_{p=\overline{p}} < 1.$$

Then, the steady state $\overline{p} = \frac{b\,(c+\gamma\,\epsilon)+\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}}{3\,b}$, of the system (2.5) is stable under the condition $|1+\alpha\,(2\,b\,(c+\gamma\,\epsilon)-6\,b\,\overline{p})| < 1$. Substituting by $\overline{p}$ in this condition, we get the condition of stability in the form;

$$0 < \alpha\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b} < 1.$$

Then, the threshold value for the speed of adjustment (the reaction coefficient) of the stability of the monopolist is;

$$\alpha < \frac{1}{\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}}.$$

We used some numerical calculations to show the changes in the behaviour of the system by varying the values of the parameter α. Let the initial values of the price is $p = 0.25$ and the values of the system parameters are $a = 3.0$, $b = 1.5$, $c = 0.3$ and $\epsilon = 0.2$, respectively. Fig. 1 shows the bifurcation diagram for the price p for different values for γ. We note that increasing the values of the parameter α changes the dynamics of the system from steady state to bifurcation state and finally the system become chaotic.

Also, increasing the values of γ decreases the values at which the behaviour of the system starts to bifurcate. This is plausible because the increasing of the variance of the random variable γ decreases the period of stability of the market. In Fig. 1 the value of γ are -0.8, -0.4, 0.0, 0.4, 0.8, 1.0 and the corresponding values of α that starts the bifurcations are 0.272, 0.271, 0.270, 0.268, 0.267, 0.266.

4 Risk in Multi-objective Model under Uncertainty

In this section, an alternative approach for risk measurement is applied. This approach has been introduced in details by Ahmed et al. [20].

Since the main aim of any firm is to maximize its profit and at the same time minimize the risk corresponding. We used the standard deviation for modeling risk in Bertrand game, $\sigma^2 = \frac{1}{2}\,(p_t - a\,\nu_t)$, where $a\,\nu_t = \frac{1}{t-1}\left(\sum_{j=1}^{t-1} p_j\right)$. According to this method, we propose the following multi-objective Bertrand dynamical system;

$$\begin{array}{l} p_{t+1} = p_t + \omega\,\frac{\partial\,\Pi_t}{\partial\,p_t} - (1-\omega)\,(p_t - a\,\nu_t), \\ a\,\nu_{t+1} = \frac{(t-1)\,a\,\nu_t + p_t}{t} \end{array}, \tag{4.1}$$

where $\omega_1 = \omega$ and $\omega_2 = 1-\omega$ are the weights assigned to different objectives (maximize the profit, minimize the risk) such that $0 \leq \omega \leq 1$.

Using Eq. (2.3) in Eq. (4.1), one gets the following two dimension system;

$$\begin{array}{l} p_{t+1} = p_t + \omega\,[a + 2\,b\,(c+\gamma\,\epsilon)\,p_t - 3\,b\,p_t^2] - (1-\omega)\,(p_t - a\,\nu_t), \\ a\,\nu_{t+1} = \frac{(t-1)\,a\,\nu_t + p_t}{t}, \end{array} \tag{4.2}$$

Proposition 4.1. *The system (4.2) has the unique equilibrium point*

$(\overline{p}_1, \overline{a\,\nu_1}) = \left(\frac{b\,(c+\gamma\,\epsilon)+\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}}{3\,b}, \frac{b\,(c+\gamma\,\epsilon)+\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}}{3\,b}\right)$
which is asymptotically stable under the condition,

$\frac{2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,\omega-1}{1-[1-2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega} < t < \frac{\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,\omega-1}{1+[1-2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega}.$

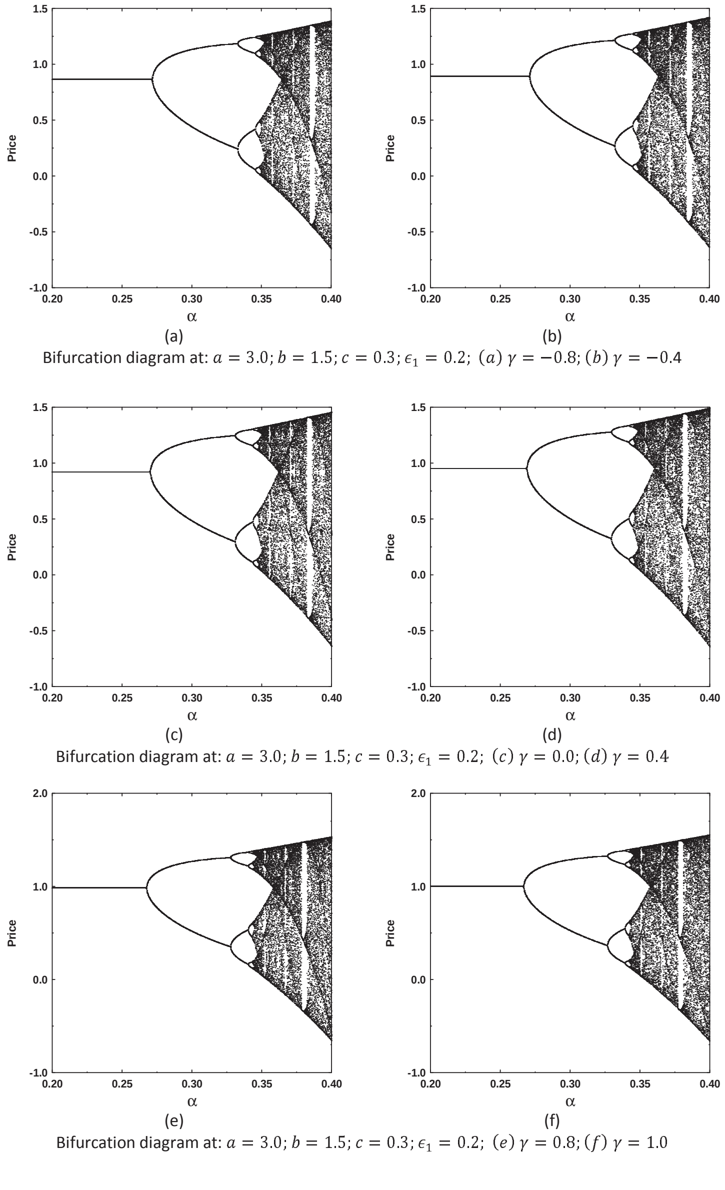

Fig. 1. The bifurcation diagram for the price at $a = 3.0$, $b = 1.5$, $c = 0.3$, $\epsilon = 0.2$ and for different values of γ

Proof To study the stability of this point, the Jacobian matrix is given as follows;

$$\begin{bmatrix} [1-\,2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega & 1-\omega \\ \frac{1}{t} & 1-\frac{1}{t} \end{bmatrix},$$

which has the characteristic equation $\lambda^2-\beta\,\lambda+\delta=0$, where

$$\beta=\left[[1-\,2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega+1-\frac{1}{t}\right]$$

and

$$\delta=[1-\,2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega+\frac{2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,\omega-1}{t},$$

Its eigenvalues $|\lambda_i|\,<\,1, i=1,2$, if the following condition is satisfied;

$$\frac{2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,\omega-1}{1-[1-2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega}\,<\,t\,<\,\frac{\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,\omega-1}{1+[1-2\,\sqrt{b^2\,(c+\gamma\,\epsilon)^2+3\,a\,b}\,]\,\omega},$$

then the equilibrium point $(\overline{p}_1,\,\overline{a\,\nu_1})$ is asymptotically stable.

5 Conclusion

In this paper, we studied the local stability of a steady state of a monopoly under uncertainly cost function. This steady state is exactly the value that maximize the profit of the firm that adapt the proposed price (2.1). We find that variation of the parameter has an important effect on the behaviour of the system. Also, we studied an approach for risk in a Bertrand game which based on multi-objective method. We will extend this work to the doubly case.

Competing Interests

The author declares that no competing interests exist.

References

[1] Bischi GI, et al. Oligopoly games with local monopolistic approximation, Journal of Economics Behavior and Organization. 2007;62:371-388.

[2] Puu T. The chaotic monopolist, Chaos, Solitons and Fractals. 1995; 5:35-44.

[3] Bischi GI, Lamantia F. Nonlinear duopoly games with positive cost externalities due to spillover effects. Chaos, Solitons and Fractals. 2002; 13:701-721.

[4] Elsadany AA. A dynamic Cournot doupoly model with different strategies. Journal of Egyptian Mathematical Society. 2015;23:56-61.

[5] Elsadany AA, et al. Complex dynamics and chaos control of heterogenous quadropoly model. Applied Mathematics and Computations. 2013;219:11110-11118.

[6] Ahmed E, et al. On Bertrand doupoly game with differentiated goods. Applied Mathematics and Computations. 2015;251:169-179.

[7] Webb J. Game Theory: Decisions, Interaction and Evolution, Springer-Verlag; 2007.

[8] Hofbauer J, Sigmund K. Evolutionary games and population dynamics. Cambridge University Press; 1998.

[9] Fudenberg D, Levine D. The theory in learning in games. Cambridge, MIT Press, 1998.

[10] Gibbons R. Game theory for applied economists. Princeton University press, US; 1992.

[11] Elaydi S. An introduction to difference equations. New York, Springer-Verlag; 2005.

[12] Abraham R, et al. Chaos in discrete dynamical systems: A visual introduction in two dimensions, Berlin, Springer; 1977.

[13] Sandmo A. On the theory of the competitive firm under price uncertainty. American Economics Review. 1971; 61:65-73.

[14] Feder G. The impact of uncertainty in a class of objective functions. Journal of Economics Theory. 1977;16:504-512.

[15] Dardononi V. Optical choice under uncertainty: the case of two-argument utility function. Economics Journal. 1988;98:429-450.

[16] Gradstein M, et al. The effects of uncertainty on interactive behavior. Economics Journal. 1992;102:554-561.

[17] Chiarella C, Szidarovszky F. Cournot oligopolies with product differentiation under uncertainly. Computational and Mathematical Applied. 2005;50:413-4240.

[18] Rockafellar RT. Coherent approaches to risk in optimization under uncertainty. Informs. 2007;139:38-61.

[19] Chiarella C, et al. Isoelastic oligopolies under uncertainly. Applied Mathematics and Computations. 2013;219:10475-10486.

[20] Ahmed E, et al. On multiobjective oligopoly. Nonlinear Dynnamical Psychology and Life Science. 2003;7:205-219.

[21] Naimzada AK, Ricchiuti G. Complex dynamics in a monopoly with a rule of thumb. Applied Mathematics and Computations. 2008;203:921-925.

[22] Askar SS. On complex dynamics of monopoly market. 2013;31:586-589.

[23] Puu T. Chaos in duopoly pricing, Chaos, Solitons and Fractals. Economics Modelling. 1991;1:573-581.

[24] Edelstein-Keshet L. Mathematical Models in Biology, SIAM; 2005.

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
http://sciencedomain.org/review-history/13094