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On the L-duality of Higher Order Finsler Spaces with (α, β)-metric

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Authors' contributions

This work was carried out in collaboration between both the authors. Author GS provided the concept and helped out determining the metrics to be worked upon. Author VS worked out the approach through calculation and managed the analyses of the study with literature searches. Both authors read and approved the final manuscript.

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Abstract

In this paper we have constructed Randers, Kropina and Matsumoto space of order-k and their L-duals respectively, using the concepts of higher order(order k) Riemmanian, Finsler, Lagrangian structures and Legendre transformation.

Keywords: Finsler space of order k; cartan space of order k; L-duality; legendre mapping.

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1 Introduction

Finsler space with (α, β)-metric deals with numerous significant metrics. Researchers on different level have worked upon, providing their beneficial and substantial results in various fields. The concept of L-duality too have been studied upon and have been worked its wonder ([1], [2], [3], [4]).

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In ([5], [6]), one can see the L-duals of Randers and Kropina metric which are quite efficient and interesting. In [7], L-dual of very famous Matsumoto metric has been determined. The L-duality of Finsler and Lagrange spaces have been brought into by R. Miron[8] and researched on large scale by other authors. The importance of L-duality is to most extent limited for computing the dual of some Finsler fundamental functions as some of the geometrical problems of (α, β)-metrics are getting resolved by considering not the metric itself, but its dual.

The concept of higher order(order k) Riemmanian, Finsler, Lagrangian structures were introduced by R. Miron([9], [10], [11]). In the present paper the L-duals of higher order(order k) Randers, Kropina and Matsumoto metric, using the definitions of higher order (α, β)-metrics and the legendre mapping, has been constructed and proved that basically their L-duals are infact the first order (α, β)-metrics.

2 Higher Order Spaces

The notion of autonomous and non-autonomous Lagrangians and Hamiltonians have been well defined, but in order to describe Lagrange space and Hamilton space of order k it is advisable for us to study the autonomous Lagrangians and Hamiltonians because the concepts of higher order (order k) spaces are the geometrical ones. These geometries can be determined over the differentiable manifolds T^kM and $T^{*k}M$, respectively. Thus we will make them as the base of our work. It's been proved that the geometries dealing with higher order Lagrange space and Hamilton space are dual and this duality is obtained via a Legendre transformation [12].

Therefore to work upon the L-duality of Finsler spaces of order k we firstly have to introduce Lagrange, Hamilton and Cartan spaces of order k.

Definition 2.1. [13]The mapping $L : T^kM \to R$ is considered as the Lagrangian of order k, $(k \in N)$, where $L = L(x, y^{(1)}, ..., y^{(k)})$ is a real valued function on T^kM, i.e., with change of local coordinates on T^kM, we have $L(\tilde{x}, \tilde{y}^{(1)}, ..., \tilde{y}^{(k)}) = L(x, y^{(1)}, ..., y^{(k)})$

A differentiable **Lagrange space** of order k is of C^∞-class on T^kM, plus it is continuous on the null section of the projection $\pi^k : T^kM \to M$.

Consider a Lagrange space of order k a pair $L^{(k)}n = (M, L)$, where M is a real n-dimensional manifold and $L : (x, y^{(1)}, ..., y^{(k)}) \in T^kM \to L(x, y^{(1)}, ..., y^{(k)}) \in R$ is a differentiable Lagrangian of order k, for which the Hessian with the entries

$$g_{ij}(x, y^{(1)}, ...y^{(k)}) = \frac{1}{2}\frac{\partial^2 L}{\partial y^{(k)i}\partial y^{(k)j}},$$

has the property

$$rank \parallel g_{ij} \parallel = n$$

Remark: [12] The pair $L^{(k)n} = (M, F^2(x, y^{(1)}, ..., y^{(k)}))$ is a Lagrange space of order k. Conversely, if $L^{(k)n} = (M, L(x, y^{(1)}, ..., y^{(k)}))$ is a Lagrange space of order k, having the fundamental function L positively, $2k$-homogeneous and the fundamental tensor g_{ij} positively defined, then the pair $F^{(k)n} = (M, \sqrt{L})$ is a Finsler space of order k.

Definition 2.2. A **Hamilton space** of order k, is the pair defined as

$$H^{(k)n} = (M, H(x, y^{(1)}, ..., y^{(k-1)}, p))$$

consisting of real n-dimensional manifold M and a regular Hamiltonian function $H : T^{*k}M \to R$ defined on the manifold $T^{*k}M$.

We consider the defined Hamilton space as differentiable Hamiltonian whose Hessian with respect to the momenta p_i, with the entries:

$$g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$$

is nondegenerate on $T^{*k}M$, with its regularity condition expressed by

$$rank\ ||g^{ij}|| = n,\ on\ T^{*k}M.$$

As a matter of fact g^{ij} here is symmetric, contravariant of order 2 and a d-tensor field.

If the base manifold M is paracompact, then the manifold $T^{*k}M$ is paracompact too, and on $T^{*k}M$ there exist regular Hamiltonians. Now we look upon the Cartan space,

Definition 2.3. The **Cartan space** of order k is a pair

$$C^{(k)n} = (M, K(x, y^{(1)}, ..., y^{(k-1)}, p))$$

for which the following axioms hold:

1. K is a real function differentiable on the manifold $T^{*k}M$ and continuous on the null section of the projection $\pi^{*k} : T^{*k}M \to M$.
2. $K > 0$ on $T^{*k}M$.
3. K is positively k-homogeneous on the fibres of bundle $T^{*k}M$, i.e.

$$K(x, ay^{(1)}, ..., a^{(k-1)}y^{(k-1)}, a^k p) = a^k K(x, y^{(1)}, ..., y^{(k1)}, p), \forall a \in R^+.$$

4. The Hessian of K^2, with respect to the momenta p_i, having the elements

$$g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p) = \frac{1}{2}\frac{\partial^2 K^2}{\partial p_i p_j}$$

is positively defined.

3 Legendre Transformation

Consider the Lagrange space of order k, $L^{(k)n} = (M, L(x, y^{(1)}, ..., y^{(k)}))$ which determines a local diffeomorphism $\varphi :\ T^k M \to T^{*k}M$ preserving the fibres. We have the following:

Proposition 3.1. [12] The mapping $\varphi : u = (y^{(0)}, y^{(1)}, ..., y^{(k)}) \in T^k M \to u^* = (x, y^{(1)}, ..., y^{(k-1)}, p) \in T^{*k}M$ given by

$$x^i = y^{(0)i}, y^{(1)i} = y^{(1)i}, ..., y^{(k-1)i} = y^{(k-1)i},$$
$$p_i = \frac{1}{2}\frac{\partial L}{\partial y^{(k)i}} \tag{3.1}$$

is a local diffeomorphism, which preserves the fibres.

Proof. The mapping φ is differentiable on the manifold $T^k M$ and its Jacobian has the determinant equal to det $||a_{ij}|| \neq 0$. Of course, $\pi^k(y^{(0)}, ..., y^{(k)}) = \pi^{*k} o\varphi(y^{(0)}, ..., y^{(k)}) = y^{(0)}$.

The diffeomorphism φ is called the Legendre mapping (or Legendre transformation).
We denote

$$p_i = \frac{1}{2}\frac{\partial L}{\partial y^{(k)i}} = \varphi_i(y^{(0)}, y^{(1)}, ..., y^{(k)}) \tag{3.2}$$

Clearly, φ_i is a d-covector field on T^kM.

The local inverse diffeomorphism $\xi = \varphi^{-1} : T^{*k}M \rightarrow T^kM$ is expressed by

$$y^{(0)i} = x^i, y^{(1)i} = y^{(1)i}, ..., y^{(k-1)i} = y^{(k-1)i}, \tag{3.3}$$

$$y^{(k)i} = \xi^i(x, y^{(1)}, ..., y^{(k-1)}, p). \tag{3.4}$$

With respect to a change of local coordinates on the manifold T^kM, ξ_i is transformed exactly as the variables $y^{(k)i}$.

The mappings φ and ξ satisfy the conditions

$$\xi o\varphi = I_{\hat{U}}, \varphi o\xi = I_{\check{U}}, \hat{U} = (\pi^{*k})^{-1}(U), \check{U} = (\pi^k)^{-1}(U), (U \in M)$$

Therefore we have the following identities

$$a_{ij}(y^{(o)}, ..., y^{(k)}) = \frac{\partial \varphi_i}{\partial y^{(k)j}};$$

$$a^{ij}(x, y^{(1)}, ..., y^{(k-1)}, \xi(x, y^{(1)}, ..., y^{(k-1)}, p)) = \frac{\partial \xi^i}{\partial p_j} \tag{3.5}$$

and

$$\frac{\partial \varphi_i}{\partial x^j} = -a_{is}\frac{\partial \xi^s}{\partial x^j}; \frac{\partial \varphi_i}{\partial y^{(\alpha)j}} = -a_{is}\frac{\partial \xi^s}{\partial y^{(\alpha)j}}, (\alpha = 1, ..., k-1); \frac{\partial \varphi_i}{\partial y^{(k)j}} = a_{ij}; \tag{3.6}$$

$$\frac{\partial \xi^i}{\partial x^j} = -a^{is}\frac{\partial \varphi_s}{\partial x^j}; \frac{\partial \xi^i}{\partial y^{(\alpha)j}} = -a^{is}\frac{\partial \varphi_s}{\partial y^{(\alpha)j}}, (\alpha = 1, ..., k-1); \frac{\partial \xi^i}{\partial p_j} = a^{ij} \tag{3.7}$$

□

Further, to achieve our results we shall use the following theorem:

Theorem 3.1. [12] The Hamiltonian function

$$H(x, y^{(1)}, ..., y^{(k-1)}, p) = 2p_i\check{z}^{(k)i} - L(x, y^{(1)}, ..., y^{(k-1)}, \xi(x, y^{(1)}, ..., y^{(k-1)}, p)), \tag{3.8}$$

is the fundamental function of a Hamilton space of order k, $H^{(k)n}$ and its fundamental tensor field is

$$g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p) = a^{ij}(x, y^{(1)}, ..., y^{(k-1)}, \xi(x, y^{(1)}, ..., y^{(k-1)}, p)),$$

a_{ij} being the fundamental tensor field of the Lagrange space of order k, $L^{(k)n} = (M, L)$.

Proof. From (3.8) we have,

$$\frac{1}{2}\frac{\partial H}{\partial p_j} = \check{z}^{(k)j} + p_m\frac{\partial \check{z}^{(k)m}}{\partial p_j} - \frac{1}{2}\frac{\partial L}{\partial y^{(k)m}}\frac{\partial \xi^{(k)m}}{\partial p_j}$$

$$= \check{z}^{(k)j} + p_m\frac{\partial \check{z}^{(k)m}}{\partial p_j} - p_m\frac{\partial \check{z}^{(k)m}}{\partial p_j} = \check{z}^{(k)j} \tag{3.9}$$

Consequently,

$$g^{ij}(x, y^{(1)}, ..., y^{(k-1)}, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j} = \frac{1}{2}\frac{\partial \check{z}^{(k)j}}{\partial p_i} = \frac{1}{2}\frac{\partial \xi^{(k)j}}{\partial p_i}$$

$$= a^{ij}(x, y^{(1)}, ..., y^{(k-1)}, \xi) \tag{3.10}$$

□

From the above statement it is clear that we have a regular Hamiltonian H with g^{ij} as its fundamental tensor having a constant signature on the manifold $T^{*k}M$. The Hamilton space of order k, $H^{(k)n} = (M, H)$ is called the dual of the Lagrange space of order k, $L^{(k)n} = (M, L)$.

In this way for Cartan and Finsler space of order k, let $F^2 = \xi^*(K^2)$ be the Lagrangian, where the mapping ξ^* is a local diffeomorphism which preserves the fibres of $T^{*k}M$ and T^kM,

$$F^2(x, y^{(1)}, ..., y^{(k)}) = 2p_i z^{(k)i}(x, y^{(1)}, ..., y^{(k-1)}, \xi^*) - K^2(x, y^{(1)}, ..., y^{(k-1)}, \varphi^*) \tag{3.11}$$

where φ^* is the local inverse of ξ^*, from which we state that

The pair (M, F), with F from (3.11), has the properties:

1. It is a Finsler space, having the fundamental function F^2, $2k$ -homogeneous on the fibres of T^kM.
2. Its fundamental tensor field is given by

$$a_{ij}(u) = g_{ij}(x, y^{(1)}, ..., y^{(k1)}, \varphi^*(u))$$

So, the space (M, F) is called the Finsler space of order k dual to the Cartan space of order k, $C^{(k)n} = (M, K)$.

4 The L-dual of an (α, β) Finsler Space of Higher Order

In this section the idea and assumption taken at the beginning of this research paper have been proved, in form of theorems.

a. The L -dual of Randers space of order-k

We consider $F^{(k)n} = (M, F)$ to be Randers space of order-k, where $F = \alpha + \beta$ is a Randers metric, in which $\alpha^2 = a_{ij}z^{(k)n}$ and $\beta = b_i(x)z^{(k)n}$. Then, we have the following:

Theorem 4.1. Let (M, F) be a Randers space of order-k and $b = \sqrt{a_{ij}b^ib^j}$ the Riemmanian length of b_i.Then:

1. If $b^2 = 1$, the L-dual of (M, F) is a Kropina space on T^*M with:

$$H(x, p) = \frac{1}{2}\left(\frac{a^{ij}b_ib_j}{2b^ip_i}\right)^2 \tag{4.1}$$

2. If $b^2 \neq 1$, the L-dual of (M, F) is a Randers space on T^*M with:

$$H(x, p) = \frac{1}{2}(\sqrt{(\bar{a}^{ij}p_ip_j} \pm \bar{b}^ip_i)^2 \tag{4.2}$$

where

$$\bar{a}^{ij} = \frac{1}{(1-b^2)}a^{ij} + \frac{1}{(1-b^2)^2}b^ib_j; \bar{b}^i = \frac{1}{(1-b^2)}b^i$$

(in(4.2) $'-'$ corresponds to $b^2 < 1$ and $'+'$ corresponds to $b^2 > 1$).

Proof. We put $\alpha^2 = y_iy^i, b^i = a^{ij}b_j, \beta = b_iy^i, \beta^* = b^ip_i, p^i = a^{ij}p_j, \alpha^{*2} = p_ip^j = a^{ij}p_ip_j$.We have,

$$F = \alpha + \beta, \quad p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(k)i}} = F(\frac{a_{ij}z^{(k)j}}{\alpha} + b_i) \tag{4.3}$$

Contracting (4.3) by p^i and b^i, we get

$$\alpha^{*2} = F(\frac{F^2}{\alpha} + \beta^*)$$
$$and\ \beta^* = F(\frac{\beta}{\alpha} + b^2)$$

Hence

$$\alpha^{*2} = F(\frac{F^2}{\alpha} + \beta^*)\ and\ \beta^* = F(\frac{\beta}{\alpha} + b^2) \tag{4.4}$$

Therefore from above equations, we have

$$\beta^* = F(\frac{F}{\alpha} + b^2 - 1) \tag{4.5}$$

Now we will define two cases: Case (1). If $b^2 = 1$, from (4.5) we obtain,

$$\beta^* = \frac{F^2}{\alpha} \tag{4.6}$$

On substituting (4.6) in the first equation of (4.4) we get,

$$\alpha^{*2} = F(2\beta)$$

which implies

$$F(x,p) = \frac{\alpha^{*2}}{2\beta^*} = \frac{a^{ij}p_ip_j}{2b^ip_i}$$

and we know that $H = \frac{1}{2}F^2$, thus giving (4.1).

Case (2). If $b^2 \neq 1$, from (4.5) and (4.6) we have,

$$\frac{1}{F}\alpha^{*2} = \left(\frac{F^2}{\alpha} + \beta^*\right) and\ \beta^* = \frac{F^2}{\alpha} + F(b^2 - 1)$$

and by substitution,

$$\frac{\beta^*}{(1-b^2)} - \frac{\alpha^{*2}}{F(1-b^2)} = -(F + \frac{\beta^*}{1-b^2})$$

which implies,

$$F = \frac{\beta^*}{b^2-1} \pm \sqrt{(\frac{\beta^*}{1-b^2})^2 + \frac{\alpha^{*2}}{1-b^2}}$$
$$\Rightarrow F = \frac{1}{b^2-1}b^ip_i \pm \sqrt{\left(\frac{1}{(1-b^2)^2}b^ib^j + \frac{1}{(1-b^2)}a^{ij}\right)p_ip_j}$$

We know that $H = \frac{1}{2}F^2$. Hence we get equation(4.2), thus proving our required result. □

b. The L-dual of a Kropina space of order-k

We consider $F^{(k)n} = (M, F)$ to be Kropina space of order-k, where $F = \dfrac{\alpha^2}{\beta}$ is a Kropina metric, in which α and β have their usual meaning as defined earlier. Then, we have the following:

Theorem 4.2. Let (M, F) be a Kropina space of order-k and $b = \sqrt{a_{ij}b^i b^j}$ the Riemmanian length of b_i. Then:

1. If $b^2 = 1$, the L-dual of (M, F) is a Randers space on T^*M with the Hamiltonian function:

$$H(x, p) = \frac{1}{2}(\bar{b}^i p_i)^2 \tag{4.7}$$

2. If $b^2 \neq 1$, The L-dual of (M, F) is a Randers space on T^*M with the Hamiltonian:

$$H(x, p) = \frac{1}{2}(\sqrt{\bar{a}^{ij} p_i p_j} \pm \bar{b}^i p_i)^2 \tag{4.8}$$

where

$$\bar{a}^{ij} = \frac{b^2}{4} a^{ij}; \bar{b}^i = \frac{1}{2} b^i$$

(Here '-' corresponds to $\beta < 0$ and '+' corresponds to $\beta > 0$).

Proof. We use same notations as in the proof of previous theorem, we have $F = \frac{\alpha^2}{\beta}$ then

$$p_i = \frac{1}{2}\frac{\partial F^2}{\partial y^{(k)i}} = F\left(\frac{2\beta\alpha\frac{y_i}{\alpha} - \alpha^2 b_i}{\beta^2}\right)$$

$$p_i = \frac{F}{\beta}(2a_{ij}z^{(k)j} - Fb_i) \tag{4.9}$$

Contracting above equation with p^i and b^i, we get

$$\alpha^{*2} = \frac{F^2}{\beta}(2F - \beta^*) \tag{4.10}$$

and

$$\beta^* = \frac{F}{\beta}(2\beta - Fb^2) \tag{4.11}$$

From here we conclude two cases:

Case (1). If $b^2 = 1$ then from (4.11) we get,

$$F = \frac{\beta^*}{2}$$

or

$$F = \frac{b^i p_i}{2}$$

and we know that $H = \frac{1}{2}F^2$, thus getting equation (4.7).

Case (2). If $b^2 \neq 1$ then on multiplying both sides of equation (4.10) by b^2, we get,

$$\alpha^{*2} b^2 = \frac{F^2}{\beta}(2F - \beta^*) b^2$$

$$\Rightarrow F = \frac{1}{2}(\beta^* \pm \alpha^* b) = (\bar{b}^i p_i \pm \ b\sqrt{\bar{a}^{ij} p_i p_j})$$

where

$$\bar{a}^{ij} = \frac{b^2}{4} a^{ij}, \ \bar{b}^i = \frac{1}{2} b^i$$

and we know that $H = \frac{1}{2}F^2$, thus giving equation (4.8), and proving our theorem. □

c. The L-dual of a Matsumoto space of order-k

We consider $F^{(k)n} = (M, F)$ to be Matsumoto space of order-k, where $F = \dfrac{\alpha^2}{\alpha - \beta}$ is a Matsumoto metric, in which α and β have their usual meaning as defined earlier. Then, we have the following:

Theorem 4.3. Let (M, F) be Matsumoto space of order k and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then

1. If $b^2 = 1$, the L-dual of (M, F) is the space having the fundamental function:

$$H(x,p) = \frac{1}{2}\left(-\frac{\beta^*}{2}\frac{\left(\sqrt[3]{\alpha^{*2}} + \sqrt[3]{(\beta^* + \sqrt{\beta^{*2} - \alpha^{*2}})^2}\right)^3}{\alpha^{*2} + (\beta^{*2} + \sqrt{\beta^{*2} - \alpha^{*2}})^2}\right)^2, \tag{4.12}$$

with $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_ip_j}$ and $\beta^* = b^ip_i$ where $\tilde{a}^{ij} = b^ib^j - a^{ij}$.

2. If $b^2 \neq 1$, the L-dual of (M, F) is the space on T^*M having the fundamental function:

$$H(x,p) = \frac{1}{2}\left(\frac{\beta^*}{16\alpha^{*2}}\frac{(-3\alpha^{*2} + \sqrt{(9+8m)\alpha^{*4} - 8\alpha^{*2}\sqrt{m\alpha^{*2}\beta^{*2}})}}{\sqrt{m\alpha^{*2}\beta^{*2}}}\right) \tag{4.13}$$

with $1 - b^2 = m$, $\alpha^* = \sqrt{\tilde{a}^{ij}(x)p_ip_j}$ and $\beta^* = b^ip_i$ where $\tilde{a}^{ij} = b^ib^j - a^{ij}$.

Proof. We will prove this theorem using same notations. Here we have,

$$F = \frac{\alpha^2}{\alpha - \beta}$$

and

$$p_i = \frac{1}{2}\frac{\dot{\partial}F^2}{\partial y^{(k)i}} = F\left[\frac{2a_{ij}z^{(k)j}}{(\alpha-\beta)} + \frac{\alpha^2 b_i - \beta a_{ij}z^{(k)j}}{(\alpha-\beta)^2}\right] \tag{4.14}$$

Contracting (4.14) by p^i and b_i

$$\begin{aligned}\alpha^{*2} = p_ip^i &= \frac{F}{(\alpha-\beta)^2}\left[a_{ij}z^{(k)j}p^i(\alpha - 2\beta) + \alpha^2 b_ip_i\right] \\ &= \frac{F}{(\alpha-\beta)^2}[F^2(\alpha-2\beta) + \alpha^2\beta^*]\end{aligned}$$

and

$$\begin{aligned}\beta^* = p_ib^i &= \frac{F}{(\alpha-\beta)^2}\left[a_{ij}z^{(k)j}b^i(\alpha - 2\beta) + \alpha^2 b^2\right] \\ &= \frac{F}{(\alpha-\beta)^2}\left[(\alpha-2\beta) + \alpha^2 b^2\right]\end{aligned}$$

Hence we get,

$$\alpha^{*2} = \frac{F}{(\alpha-\beta)^2}[F^2(\alpha-2\beta) + \alpha^2\beta^*] \tag{4.15}$$

and

$$\beta^* = \frac{F}{(\alpha-\beta)^2}\left[(\alpha-2\beta) + \alpha^2 b^2\right] \tag{4.16}$$

In [14],for a Finsler (α, β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0, b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s)$, $s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $||\beta||_x < b_0, \forall x \in M$,$\phi$ satisfies $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, (|s| \leq b_0)$.

A Matsumoto metric is a special (α, β)-metric with $\phi = \frac{1}{1-s}$.

Using Shen's [14] notation $s = \frac{\beta}{\alpha}$ above equations become:

$$\alpha^{*2} = F^2 \frac{1-2s}{(1-s)^3} + F\frac{1}{(1-s)^2}\beta^* \tag{4.17}$$

$$\beta^* = Fs\frac{1-2s}{(1-s)^2} + F\frac{1}{(1-s)^2}b^2 \tag{4.18}$$

Now we put $1 - s = t$, so $s = 1 - t$ and both equations become:

$$\alpha^{*2} = F^2 \left(\frac{2t-1}{t^3}\right) + F\frac{1}{t^2}\beta^* \tag{4.19}$$

$$\beta^* = F(1-t)\left(\frac{2t-1}{t^2}\right) + F\frac{1}{t^2}b^2 \tag{4.20}$$

Now from (4.20) we have,

$$\beta^* t^2 = F(-2t^2 + 3t + b^2 - 1)F \tag{4.21}$$

From here we get two cases: Case (1). For $b^2 = 1$ from (4.21) we get,

$$F = -\frac{\beta^* t}{2t-3} \tag{4.22}$$

On substituting value of F from (4.22) in (4.19) we get,

$$\alpha^{*2} = \left(-\frac{\beta^* t}{2t-3}\right).\frac{2t-1}{t^3} + \left(\frac{-\beta^* t}{2t-3}\right).\frac{1}{t^2}.\beta^*$$

$$\Rightarrow t^3 - 3t^2 + \frac{9}{4}t - \frac{\beta^{*2}}{2\alpha^{*2}} = 0$$

Using Cardano's method for solving above cubic equation, we have,

$$t = 1 + P + Q$$

where

$$P = \left(\frac{1}{2}\right) Z^{\frac{2}{3}},\ Z := \frac{[\beta^* + (\beta^{*2} - \alpha^{*2})^{\frac{1}{2}}]}{\alpha} \tag{4.23}$$

$$Q = \left(\frac{1}{2}\right) W^{\frac{2}{3}},\ W := \frac{[\beta^* - (\beta^{*2} - \alpha^{*2})^{\frac{1}{2}}]}{\alpha} \tag{4.24}$$

We assume $(\beta^{*2} - \alpha^{*2})$ is positive and β^* is positive, and $W = \frac{1}{Z}$, we obtain,

$$F = \frac{-\beta^* t}{2t-1} = -\beta^* \frac{(1+P+Q)}{2+2P+2Q-3} \tag{4.25}$$

On putting values of P and Q in (4.25) we get,

$$F = -\frac{\beta^*}{2}\frac{(Z^{\frac{2}{3}}+1)^3}{(Z^{\frac{2}{3}}-1)^3} \tag{4.26}$$

On substituting value of Z from (4.23) we have (4.12).

Case (2). If $b^2 \neq 1$ (4.21) gives

$$F = \frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}, \tag{4.27}$$

and putting this in (4.19) we get,

$$\alpha^{*2} = \left(\frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}\right)^2 \left(\frac{2t-1}{t^3}\right) + \frac{\beta^* t^2}{-2t^2 + 3t + b^2 - 1}\frac{1}{t^2}\beta^*$$

$$\Rightarrow t^4 - 3t^3 + t^2\frac{13-4b^2}{4} + t\frac{6\alpha^{*2}(b^2-1) - 2\beta^{*2}}{4\alpha^{*2}} + \frac{\alpha^{*2}(b^2-1) + \beta^{*2}(1-b^2)}{4\alpha^{*2}} = 0 \tag{4.28}$$

which is a biquadratic equation, providing four roots of t given by:

$$t_1 = \frac{3}{4} - \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\mu - \frac{\eta}{4\lambda}}$$
$$t_2 = \frac{3}{4} - \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\mu - \frac{\eta}{4\lambda}}$$
$$t_3 = \frac{3}{4} + \frac{1}{2}\lambda - \frac{1}{2}\sqrt{\mu + \frac{\eta}{4\lambda}}$$
$$t_4 = \frac{3}{4} + \frac{1}{2}\lambda + \frac{1}{2}\sqrt{\mu + \frac{\eta}{4\lambda}}$$

where

$$\lambda = \sqrt{\frac{9}{4} + \frac{1}{4}(-13+4b^2) - \frac{-13\alpha^{*2} + 4b^2\alpha^{*2}}{12\alpha^{*2}} - \frac{p}{(6\times 2^{1/3})q^{1/3}} - \frac{1}{(12\times 2^{1/3})q^{1/3}}},$$

$$p = -95\alpha^{*2} + 160b^2\alpha^{*2} + 16b^4\alpha^{*2} - 24\beta^{*2} - 48b^2\beta^{*2},$$

$$\begin{aligned} q = {} & (2(-13\alpha^{*2} + 4b^2\alpha^{*2})^3 + 216\alpha^{*2}(-13\alpha^{*2} + 4b^2\alpha^{*2})(-3\alpha^{*2} + 3b^2\alpha^{*2} - \beta^{*2}) - 432\alpha^{*2}(-3\alpha^{*2} + 3b^2\alpha^{*2} \\ & - \beta^{*2}) + 3888\alpha^{*4}(\alpha^{*2} - b^2\alpha^{*2} - \beta^{*2} + b^2\beta^{*2}) + 288\alpha^{*2}(-13\alpha^{*2} + 4b^2\alpha^{*2})(\alpha^{*2} - b^2\alpha^{*2} - \beta^{*2} + b^2\beta^{*2}) \\ & + (-4(-95\alpha^{*4} + 160b^2\alpha^{*4} + 16b^4\alpha^{*4} - 24\alpha^{*2}\beta^{*2} - 48b^2\alpha^{*2}\beta^{*2})^3 + (2(-13\alpha^{*2} + 4b^2\alpha^{*2})^3 \\ & + 216\alpha^{*2}(-13\alpha^{*2} + 4b^2\alpha^{*2})(-3\alpha^{*2} + 3b^2\alpha^{*2} - \beta^{*2}) - 432\alpha^{*2}(-3\alpha^{*2} + 3b^2\alpha^{*2} - \beta^{*2})^2 \\ & + 3888\alpha^{*4}(\alpha^{*2} - b^2\alpha^{*2} - \beta^{*2} + b^2\beta^{*2}) + 288\alpha^{*2}(-13\alpha^{*2} + 4b^2\alpha^{*2})(\alpha^{*2} - b^2\alpha^{*2} - \beta^{*2} + b^2\beta^{*2}))^2)^{1/2}) \end{aligned}$$

$$\mu = \sqrt{\frac{9}{2} + \frac{1}{4}(-13+4b^2) + \frac{-13\alpha^{*2} + 4b^2\alpha^{*2}}{12\alpha^{*2}} + \frac{p}{(6\times 2^{1/3})q^{1/3}} + \frac{1}{(12\times 2^{1/3})q^{1/3}}},$$

$$\eta = 27 - 3(13 - 4b^2) - \frac{4(-3\alpha^{*2} + 3b^2\alpha^{*2} - \beta^{*2})}{\alpha^{*2}}.$$

On using software Wolfram Mathematica 8.0 and equation (4.28) we get our desired equation (4.13). Thus proving our required result. □

Conclusion

Thus its been concluded that with the help of Legendre transformation, Lagrange space, Hamilton space and Cartan space of higher order and using defined Randers, Kropina and Matsumoto metric, we have constructed the L-duals of these metrics in order-k respectively.

Competing Interests

The authors declare that they have no competing interests.

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